Quantum Mechanics: A Different Spin on Linear Algebra

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Motivation

* Quantum mechanics first formulated as “matrix mechanics”
* Students enjoy tackling “real problems” even if they don’t know all the details
* Can combine as many (or few) concepts as wanted in a single problem
* Gives a concrete interpretation to evalues

Truth in Advertising

* The class taught was "mathematical physics"
* All students were math or physics majors
* Multivariable calculus and DEs are co-requisites; a few had taken linear algebra
* First half of the course was on linear algebra

Quantum Mechanics for the Ph.D. Mathematician

1. Physical states are represented by vectors in a complex Hilbert space
2. Physical observables are represented by self-adjoint operators, and measurements correspond to their spectral projectors
3. ∃ 1-parameter unitary group of evolution

   operators. Its self-adjoint generator is called the Hamiltonian (eigenvectors are "energy levels"; eigenvalues are "energies")

Quantum Mechanics for Linear Algebra Students

1. Physical states are represented by vectors in a complex vector space
2. Physical observables are represented by Hermitian operators
3. Time evolution is given by the Schrödinger equation:

$$\psi(t) = e^{-\left(\frac{iHt}{\hbar}\right)}\psi(0)$$

   H is called the Hamiltonian, and its eigenvalues are the energies of the system

Use the ~~Force~~ Spin, Luke!

* Position and momentum operators always act on L^2–not linear algebra!
* "Angular momentum" or "spin" is a physical property possessed by virtually all objects.
* Familiar to students in the physical sciences and many others
* Other choices possible, but less familiar: quark flavor, neutrino generation, color

All About Spin

* Depending on context, spin may be called L, S, or $J = L + S$
* "Spin" is always an integer multiple of $\hbar/2$
* A vector in $\mathbb{C}^{k+1}$ represents the state of an object with spin $k\hbar/2$

Creating Problems

* Electric and magnetic fields affect the energy of and are used to manipulate spin states
* The Hamiltonian is usually a multiple of the spin operator
* Representative numbers:

Atomic energies	5×10^{-5} eV	8×10^{-24} J
nuclear energies	500 keV	8×10^{-14} J
$\hbar = h/2\pi$	6.585×10^{-16} eV·s	1.055×10^{-34} J·s

Example 1: Diagonalization

* A spin-1 nucleus moves through a complicated magnetic field where its interaction Hamiltonian is given by

$$H = E_0 \begin{pmatrix} 37 & 14i & 8 \\ -14i & 16 & -4i \\ 8 & 4i & 46 \end{pmatrix}$$

At t=0 it is in the state $\psi = (1,0,0)$, corresponding to a $L_z = +\hbar$ state. Find the state at t=2.

Example 2: Eigenvectors

* A spin-1 atoms moves through a complicated electric field where its interaction Hamiltonian is given by

$$H = E_0 \begin{pmatrix} 37 & 14i & 8 \\ -14i & 16 & -4i \\ 8 & 4i & 46 \end{pmatrix}$$

Find a vector representing the ground state.

Example 3: Multiplicity

* The interaction of the spins of the proton and electron in a Hydrogen atom is given by "fine-structure" Hamiltonian

$$H = 2.35 \times 10^{-25}\ \mathrm{J} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Determine if there are any degenerate states in this system.

Example 4: Commutator

* According to Heisenberg's Uncertainty Principle, two quantities can be simultaneously determined if and only if the corresponding operators commute. The operators for the x, y, and z components of a spin-1/2 particle like the electron are given by

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Determine whether any two components of spin can be simultaneously observed.

Example 5: The Über Problem

(Double Value Problem) A stream of spin-1 atoms is shot through a uniform magnetic field pointing in the y-direction. In a basis where the vector $\psi_1 = (1, 0, 0)$ represents the $L_z = \hbar$ state, $\psi_0 = (0, 1, 0)$ represents the $L_z = 0$ state, and $\psi_{-1} = (0, 0, 1)$ represents the $L_z = -\hbar$ state, the Hamiltonian of the system can be written:

$$H = E_0 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

(The energy E_0 is half of the magnetic dipole energy difference between the $L_y = +\hbar$ and the $L_y = -\hbar$ states.)

(a) Compute the matrix e^{iHt} which takes $\psi(0)$ to $\psi(t)$.

(b) Let $\psi(0)$ equal, in turn, each of the above basis vectors. Find the fraction of the atoms which remain in the original state after spending a time t in the magnetic field. According to the laws of quantum mechanics, that fraction is given by $|\langle\psi(0) \mid \psi(t)\rangle|^2$.

Student Response

* "I really liked that nucleus problem. I just tuned out [proof] problems ..."
* "[Schrödinger equation problems] are cool stuff"
* Students found the problems difficult at first but came to enjoy them

Further Reading

* Townsend, John. A modern approach to quantum mechanics. McGraw-Hill. New York (1992).
* Shankar, R. Principles of quantum mechanics, 2nd ed. Plenum. New York. (1994)

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