Quantum Mechanics: A Different Spin on Linear Algebra

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Motivation

- * Quantum mechanics first formulated as "matrix mechanics"
- * Students enjoy tackling "real problems" even if they don't know all the details
- * Can combine as many (or few) concepts as wanted in a single problem
- * Gives a concrete interpretation to evalues

Truth in Advertising

- * The class taught was "mathematical physics"
- * All students were math or physics majors
- Multivariable calculus and DEs are corequisites; a few had taken linear algebra
- First half of the course was on linear algebra

Quantum Mechanics for the Ph.D. Mathematician

- 1. Physical states are represented by vectors in a complex Hilbert space
- 2. Physical observables are represented by self-adjoint operators, and measurements correspond to their spectral projectors
- 3. ∃ 1-parameter unitary group of evolution operators. Its self-adjoint generator is called the Hamiltonian (eigenvectors are "energy levels"; eigenvalues are "energies")

Quantum Mechanics for Linear Algebra Students

- 1. Physical states are represented by vectors in a complex vector space
- 2. Physical observables are represented by Hermitian operators
- 3. Time evolution is given by the Schrödinger equation: $\Psi(t) = e^{-\left(\frac{iHt}{\hbar}\right)}\Psi(0)$

H is called the Hamiltonian, and its eigenvalues are the energies of the system

Use the Force Spin, Luke!

- * Position and momentum operators always act on L²-not linear algebra!
- * "Angular momentum" or "spin" is a physical property possessed by virtually all objects.
- * Familiar to students in the physical sciences and many others
- * Other choices possible, but less familiar: quark flavor, neutrino generation, color

All About Spin

* Depending on context, spin may be called L, S, or J = L + S

* "Spin" is always an integer multiple of $\hbar/2$

* A vector in \mathbb{C}^{k+1} represents the state of an object with spin $k\hbar/2$

Creating Problems

- Electric and magnetic fields affect the energy of and are used to manipulate spin states
- * The Hamiltonian is usually a multiple of the spin operator
- * Representative numbers:

Atomic energies	5 x 10 ⁻⁵ eV	8 x 10 ⁻²⁴ J
nuclear energies	500 keV	8 x 10 ⁻¹⁴ J
$\hbar = h/2\pi$	6.585 x 10 ⁻¹⁶ eV·s	1.055 x 10 ⁻³⁴ J·s

Example 1: Diagonalization

* A spin-1 nucleus moves through a complicated magnetic field where its interaction Hamiltonian is given by

 $H = E_0 \begin{pmatrix} 37 & 14i & 8 \\ -14i & 16 & -4i \\ 8 & 4i & 46 \end{pmatrix}$

At t=0 it is in the state $\psi = (1,0,0)$, corresponding to a $L_z = +\hbar$ state. Find the state at t=2.

Example 2: Eigenvectors

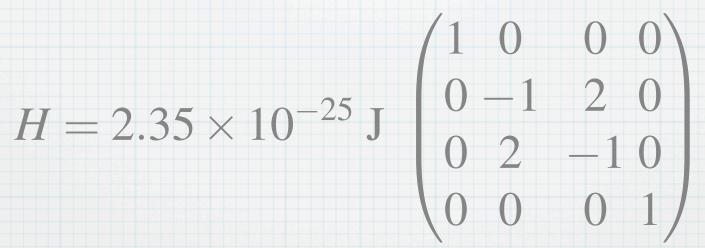
* A spin-1 atoms moves through a complicated electric field where its interaction Hamiltonian is given by

$$H = E_0 \begin{pmatrix} 37 & 14i & 8 \\ -14i & 16 & -4i \\ 8 & 4i & 46 \end{pmatrix}$$

Find a vector representing the ground state.

Example 3: Multiplicity

* The interaction of the spins of the proton and electron in a Hydrogen atom is given by "fine-structure" Hamiltonian



Determine if there are any degenerate states in this system.

Example 4: Commutator

* According to Heisenberg's Uncertainty Principle, two quantities can be simultaneously determined if and only if the corresponding operators commute. The operators for the x, y, and z components of a spin-1/2 particle like the electron are given by

Determine whether any two components of spin can be simultaneously observed.

 $S_{x} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_{y} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} S_{z} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Example 5: The über Problem

(Double Value Problem) A stream of spin-1 atoms is shot through a uniform magnetic field pointing in the y-direction. In a basis where the vector $\psi_1 = (1, 0, 0)$ represents the $L_z = \hbar$ state, $\psi_0 = (0, 1, 0)$ represents the $L_z = 0$ state, and $\psi_{-1} = (0, 0, 1)$ represents the $L_z = -\hbar$ state, the Hamiltonian of the system can be written:

$$H = E_0 \left(egin{array}{ccc} 0 & -i & 0 \ i & 0 & -i \ 0 & i & 0 \end{array}
ight)$$

(The energy E_0 is half of the magnetic dipole energy difference between the $L_y = +\hbar$ and the $L_y = -\hbar$ states.)

- (a) Compute the matrix e^{iHt} which takes $\psi(0)$ to $\psi(t)$.
- (b) Let $\psi(0)$ equal, in turn, each of the above basis vectors. Find the fraction of the atoms which remain in the original state after spending a time t in the magnetic field. According to the laws of quantum mechanics, that fraction is given by $|\langle \psi(0) | \psi(t) \rangle|^2$.

Student Response



- * "I really liked that nucleus problem. I just tuned out [proof] problems ..."
- * "[Schrödinger equation problems] are cool stuff"
- * Students found the problems difficult at first but came to enjoy them

Further Reading

- Townsend, John. <u>A modern approach to</u> <u>quantum mechanics.</u> McGraw-Hill. New York (1992).
- * Shankar, R. <u>Principles of quantum</u> <u>mechanics, 2nd ed.</u> Plenum. New York. (1994)
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