Using 3-Dimensional Bifurcation Diagrams to Enhance Student Learning

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Joint Mathematics Meeting
January 8, 2009
Washington, D.C.

What is a bifurcation?

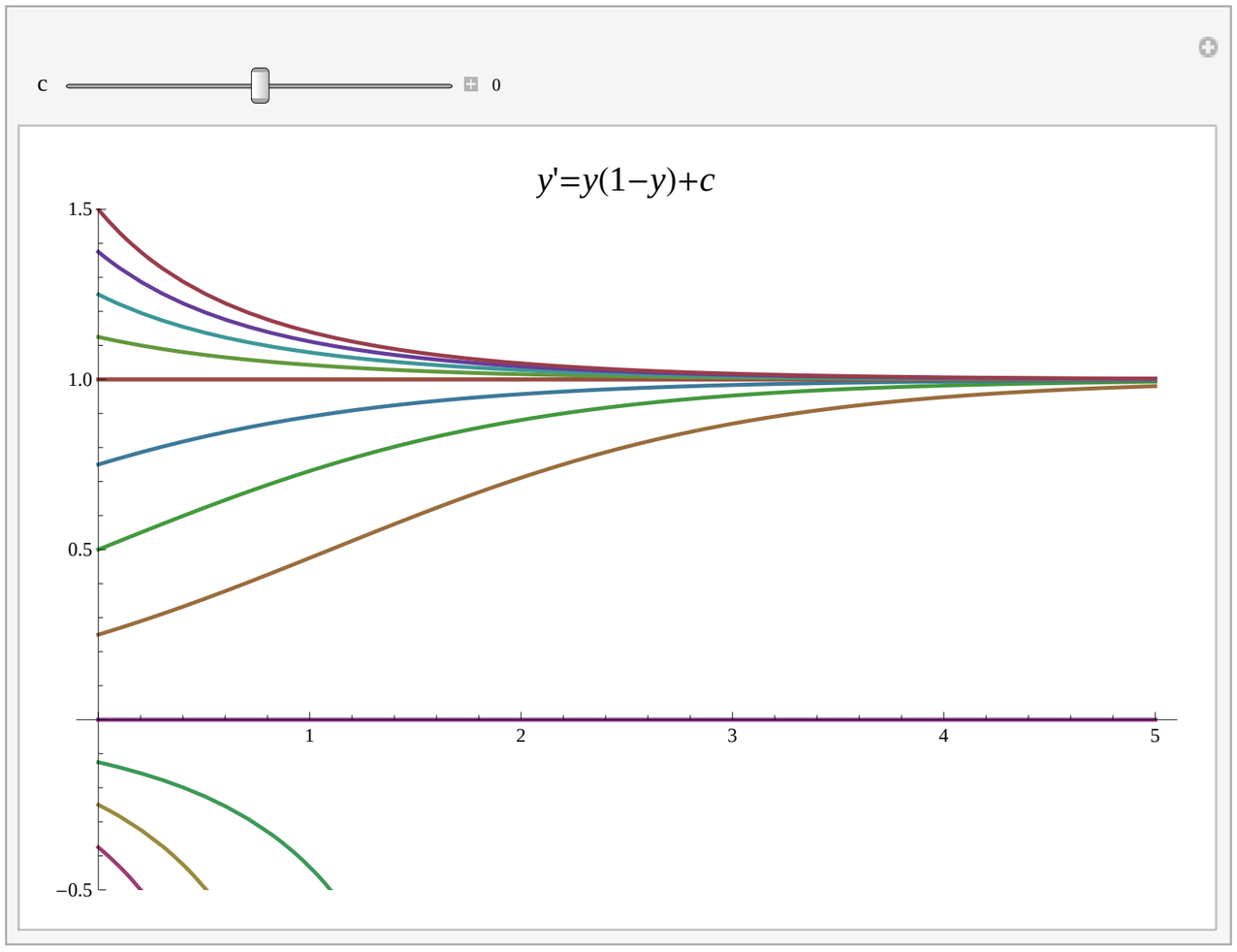
■ Bifurcation

Bifurcation: a small change in a parameter in an ODE qualitatively changes the structure of solutions.

$$y' = f(y; c)$$

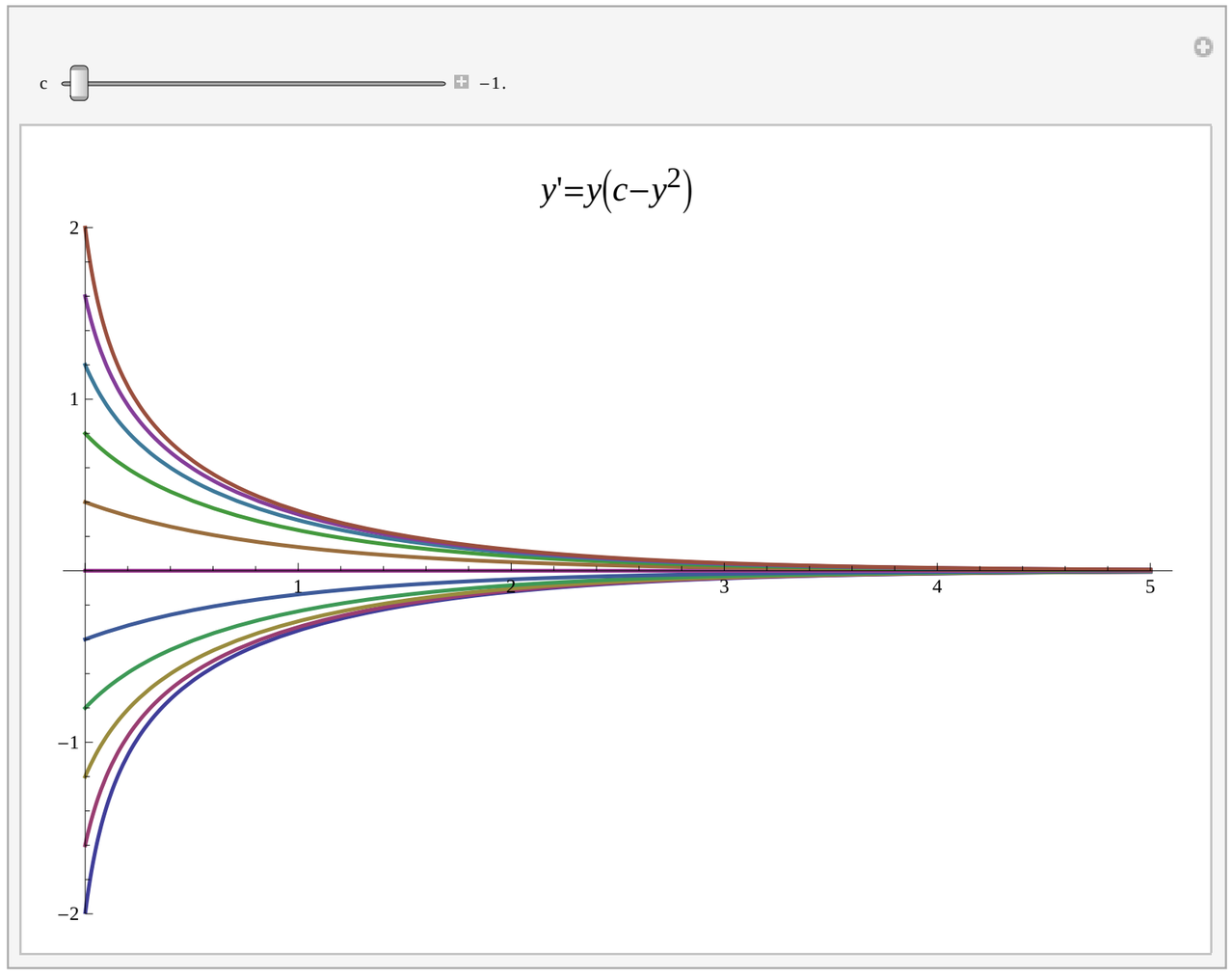
- y: dependent variable
- $y' = \frac{dy}{dt}$: first derivative
- c: a free parameter

■ Saddle-Node Bifurcation



A *saddle-node bifurcation* is a bifurcation in which *two* equilibria "annihilate." In other words, a repellor and attractor merge, resulting in the number of equilibria reduced by two.

■ Pitchfork Bifurcation



A *pitchfork bifurcation* is a bifurcation in which *three* equilibria merge. The outer two equilibria must be of opposite type to the middle one. After (or before, depending on your point of view) the bifurcation, only the middle equilibrium will remain, but it will then have opposite character.

Traditional Bifurcation Diagrams

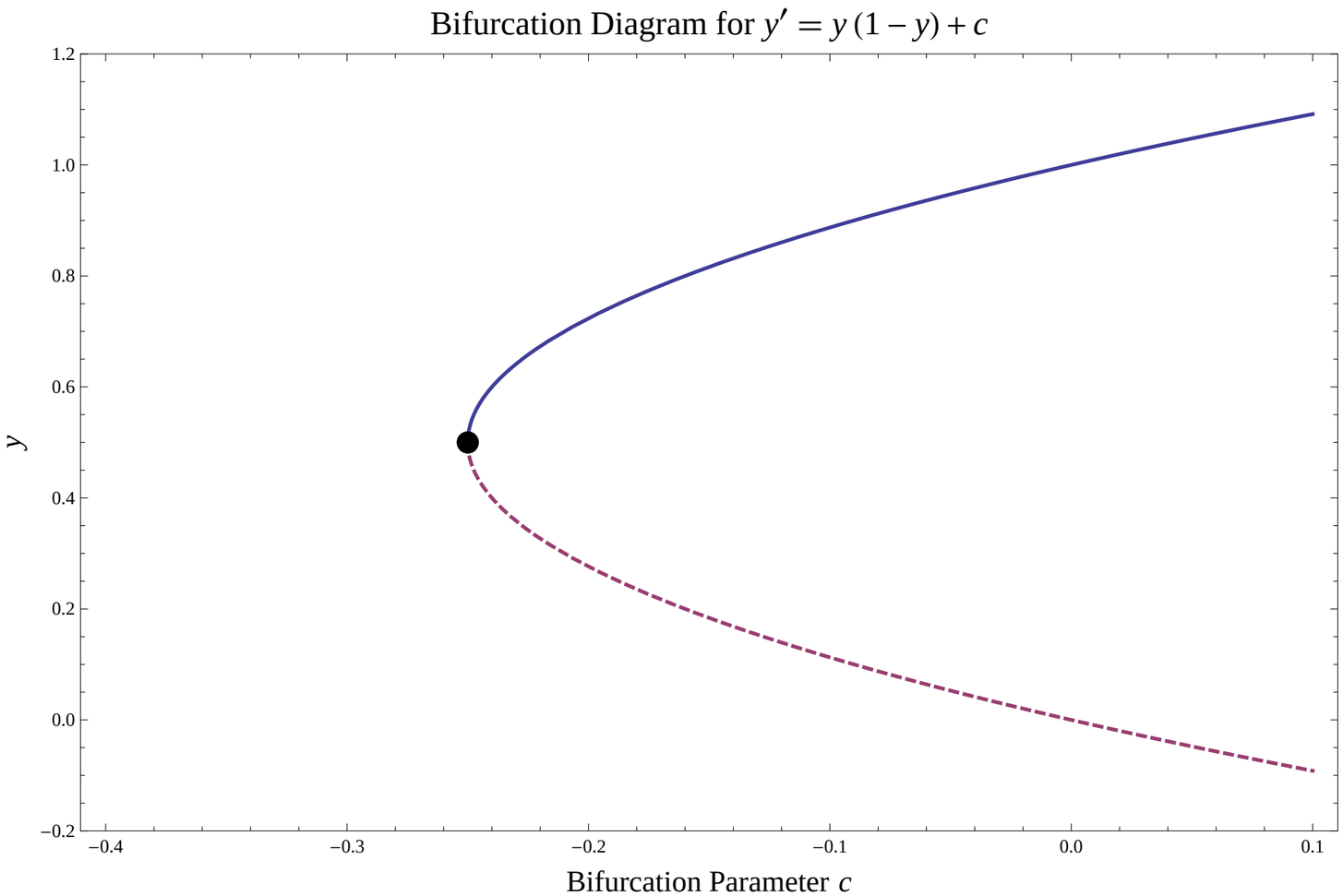
■ What is a Bifurcation Diagram?

- Contour plots of $f(y;\ c) == 0$
- Represent equilibria
- Stable are solid curves, unstable dashed curves

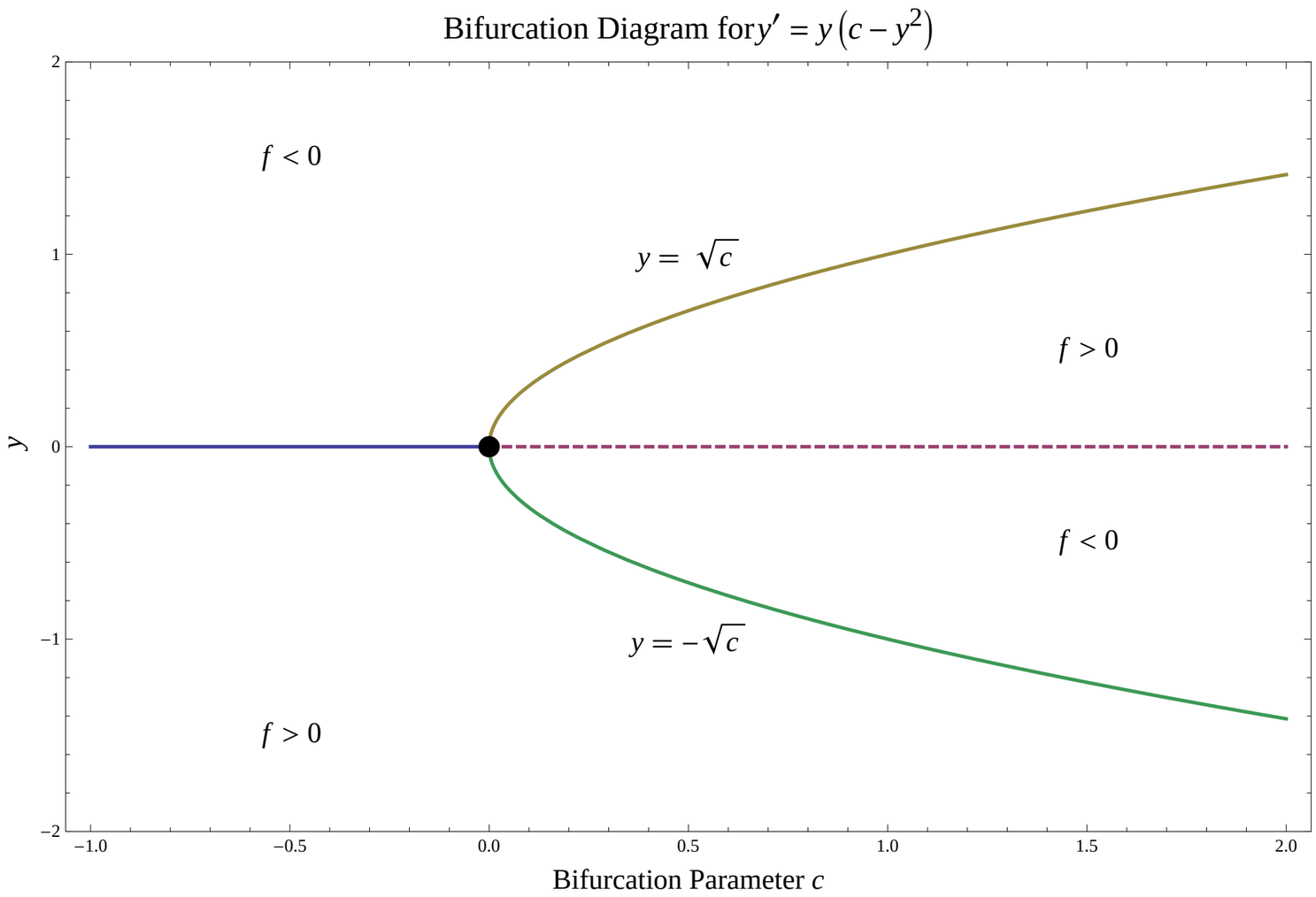
■ Pros and Cons

- Advantages
  - Relatively easy to sketch/plot
  - Some names related to shape of diagram
- Disadvantages
  - Must classify equilibria prior to creation
  - Give no information as to why a bifurcation occurs

■ Example: Saddle-Node Bifurcation



■ Example: Pitchfork Bifurcation



3-Dimensional Diagrams

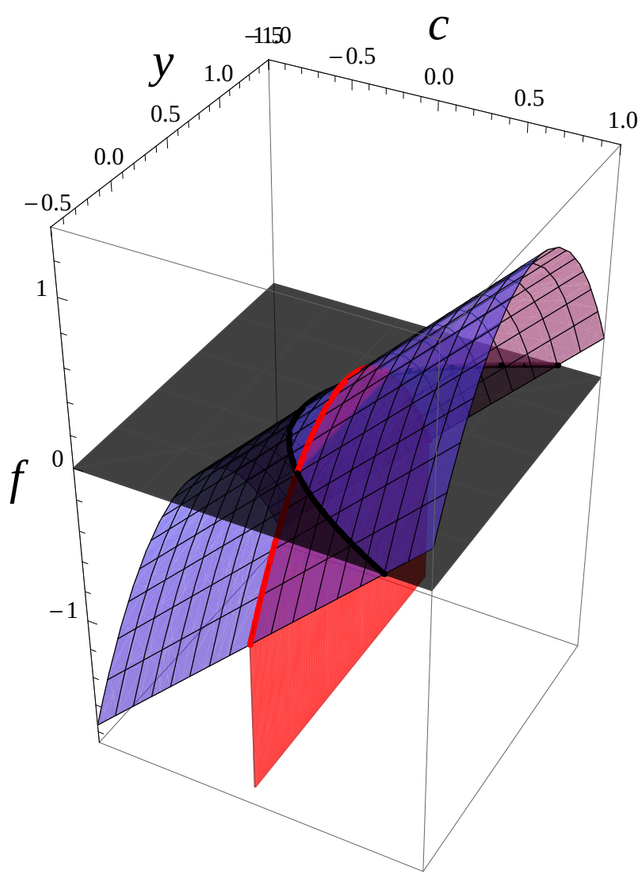
■ What Do I Mean by a 3-Dimensional Bifurcation Diagram?

- The graph of $f(y; c)$ together with the $f = 0$ section and zero or more $c =$ constant sections.
- $f = 0$ represents equilibria
- $c =$ constant section represents gives the "dynamical vector field" at constant $c \Rightarrow$ indicates stability of equilibria.

■ Fancy 3-Dimensional Diagram for Saddle-Node Bifurcation

Note: the interactive version requires a license for Knox Packages.
Just execute the cell if you have Knox Pacakges.
I include a sample image using $c == 0$.

```
Manipulate[PlotSection[y (1 - y) + x, {x, -1, 1}, {y, -0.5, 1.5},
  XSection → d, XSectionStyle -> Red, ZSection -> 0, ZSectionStyle -> Black,
  BoxRatios -> Automatic, AxesLabel -> (Style[#, Large] & /@ {c, y, f}),
  LabelStyle -> Medium],
 {{d, 0, Style[c, "TraditionalForm", 18, FontFamily → "Times"]},
  -1, 1, Appearance -> "Labeled"}]
```



■ Pros and Cons

- Advantages
  - Does not require significant analysis to create
  - Clearly shows why bifurcation occurs
  - Can be used to determine the nature of equilibria
- Disadvantages
  - Requires a powerful plotting program
  - Names of equilibria may not be as clear
  - May require significant computing power (don't create an interactive version in *Mathematica* 6)

■ A Slightly Simpler 3-D Diagram for Saddle-Node Bifurcation

Note: this code should work with *Mathematica* ≥ 6.0. Just execute the cell to see the result. *Mathematica* ≥ 7.0 is recommended.

```
Manipulate[
    Show[
        Plot3D[y(1-y)+x, {x, -1,1}, {y, -0.5,1.5},
            BoxRatios->Automatic,
            AxesLabel->(Style[#,Large]& /@ {ToExpression["c"],y,f})
        ],
        ContourPlot3D[z==0,{x, -1,1}, {y, -0.5,1.5}, {z,-2,2},
            ContourStyle->{Black, Opacity[0.5]}
        ],
        ContourPlot3D[x==c,{x, -1,1}, {y, -0.5,1.5}, {z,-2,2},
            ContourStyle->{Red, Opacity[0.5]},
            RegionFunction->(#3< #2(1-#2)+#1&)
        ]
    ],
    {{c,0}, -1,1, Appearance->"Labeled"}
]
```

■ A Basic 3-Dimensional Diagram for Saddle-Node Bifurcation

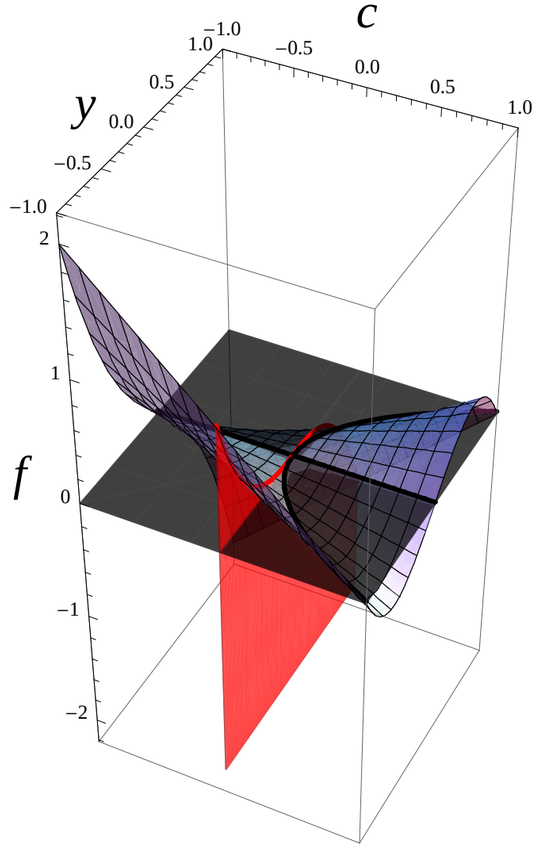
Note: this code should work with *Mathematica* ≥ 6.0. Just execute the cell to see the result. *Mathematica* ≥ 7.0 is recommended.

```
Manipulate[
    Show[
        Plot3D[y(1-y)+x, {x, -1,1}, {y, -0.5,1.5},
            BoxRatios->Automatic, AxesLabel-> {"c",y,f},
            PlotRange->{-2,1.5}
        ],
        ContourPlot3D[{z==0,x==c}, {x, -1,1}, {y, -0.5,1.5},
            {z,-2,2}, ContourStyle->{{Black, Opacity[0.75]},
                {Red,Opacity[0.75]}}
            ]
    ],
    {{c,0}, -1, 1,Appearance->"Labeled"}
]
```

■ Fancy 3-Dimensional Diagram for Pitchfork Bifurcation

Note: the interactive version requires a license for Knox Packages.
Just execute the cell if you have Knox Pacakges.
I include a sample image using $c == 0$.

```
Manipulate[PlotSection[y (x - y^2), {x, -1, 1}, {y, -1, 1},
  XSection -> d, ZSection -> 0, BoxRatios -> Automatic, Fudge -> 0.02,
  XSectionStyle -> Red, ZSectionStyle -> Black, PlotRange -> All,
  AxesLabel -> (Style[#, Large] & /@ {c, y, f})],
 {{d, 0, Style[c, "TraditionalForm", 18, FontFamily -> "Times"]},
  -1, 1, Appearance -> "Labeled"}]
```



Reception

■ Student Reception

- Positive
  - Students could create such diagrams on their own, even on timed exams
  - Students could clearly explain how the diagram indicated the type of bifurcation
  - Some stated directly that these "made a lot more sense than the book"
  - They liked the pretty pictures
- Negative
  - Very weak students have trouble creating their own graphs (especially interactive ones)

■ Faculty Reception

- "I never thought to use PlotSection in DE's, but that's a really nice application" (Dennis Schneider)
- "Ooooh, that's pretty!" (name withheld to protect reputation)