The Cauchy Problem for the Wave Operator(s)

Fun Excursions in Applied Analysis

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Theme: Naco or Anti-Naco?

Definition (Naco)

Naco: Ron Stoppable's invention. "Half nacho, half taco, all delicious". Antonym: mathematical physics.

Definition (Mathematical Physics)

Mathematical Physics: Half math, half physics, not at all delicious.

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Definition (Mathematical Physics—Correct)

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In the Beginning

If we turn on a light bulb, how does the light spread out?

If we throw a rock into a pond, what ripples will we see?

These and other physical processes are described by wave equations. We will try to understand how mathematicians deal with wave equations by analyzing one of the simplest: the classical or flat-space wave equation. We will then describe what changes for more complicated equations.

Statement of the Cauchy Problem

Definition (D'Alembertian)

AKA the classical wave operator in three-dimensions is

$$\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 = -\partial_t^2 + \triangle = -\partial_t^2 + \nabla^2.$$

Definition (Cauchy Problem)

Find a function $F(\vec{x}, t)$ which obeys

1. $\Box F = 0$ in $\mathbb{R}^4$.
2. $F(\vec{x}, 0) = f(\vec{x}) \, \forall \, \vec{x} \in \mathbb{R}^3$.
3. $\frac{\partial F}{\partial t}(\vec{x}, 0) = g(\vec{x}) \, \forall \, \vec{x} \in \mathbb{R}^3$.

The Fourier Transform

Definition (Fourier transform)

The Fourier transform of a function $f(\vec{x})$ is a function $\hat{f}(\vec{k})$ given by the formula

$$\hat{f}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d^3x.$$

This transform is invertible:

$$f(\vec{x}) = (\hat{f})^{\vee}(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{f}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} d^3k.$$

Fourier Transform and Derivatives

The Fourier transform changes derivatives into multiplication:

$$
\begin{aligned}
\widehat{(\partial_x f)}(\vec{k}) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (\partial_x f(\vec{x}))\, e^{-i\vec{k}\cdot\vec{x}} d^3x \\
&= -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(\vec{x}) \left(\partial_x e^{-i\vec{k}\cdot\vec{x}}\right) d^3x \\
&= (-)(-ik_x)\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d^3x \\
&= ik_x \hat{f}(\vec{k}).
\end{aligned}
$$

Solution of the Cauchy Problem

Theorem

The solution of the Cauchy problem is

$$F(\vec{x}, t) = \left(\hat{f}(\vec{k}) \cos \omega t + \hat{g}(\vec{k}) \frac{\sin \omega t}{\omega} \right)^{\vee},$$

where $\omega = \sqrt{k^2} = \sqrt{\vec{k} \cdot \vec{k}}$.

Proof of the Theorem

Need to check three things:

1. Does $F(\vec{x}, t)$ have the right initial value? Yes:

$$\begin{aligned} F(\vec{x}, 0) &= \left(\hat{f}(\vec{k}) \cos(\omega \cdot 0) + \hat{g}(\vec{k}) \frac{\sin(\omega \cdot 0)}{\omega}\right)^{\check{}} \\ &= \left(\hat{f}(\vec{k}) \cdot 1 + 0\right)^{\check{}} = f(\vec{x}). \end{aligned}$$

2. Does $F(\vec{x}, t)$ have the right initial derivative? Yes:

$$\begin{aligned} \frac{\partial F}{\partial t}(\vec{x}, 0) &= \left(\hat{f}(\vec{k})(-\omega \sin(\omega \cdot 0)) + \hat{g}(\vec{k}) \frac{\omega \cos(\omega \cdot 0)}{\omega}\right)^{\check{}} \\ &= \left(0 + \hat{g}(\vec{k}) \cdot 1\right)^{\check{}} = g(\vec{x}). \end{aligned}$$

Proof of the Theorem, II

3. Does F obey the wave equation? Yes. Notice

$$\widehat{\nabla F}(\vec{k},t) = i\vec{k}\hat{F}(\vec{k},t), \quad \text{and}$$

$$\widehat{\Box F}(\vec{k},t) = (-\partial_t^2 + \nabla\cdot\nabla F)\hat{}(\vec{k}) = (-\partial_t^2 - k^2)\hat{F}(\vec{k},t).$$

Now,

$$\begin{aligned} -\partial_t^2\hat{F}(\vec{k},t) &= -\partial_t^2\left(\hat{f}(\vec{k})\cos\omega t + \hat{g}(\vec{k})\frac{\sin\omega t}{\omega}\right) \\ &= (-1)^2\omega^2\left(\hat{f}(\vec{k})\cos\omega t + \hat{g}(\vec{k})\frac{\sin\omega t}{\omega}\right) \\ &= k^2\hat{F}(\vec{k},t). \end{aligned}$$

Thus $\widehat{\Box F}(\vec{k},t) = (k^2 - k^2)\hat{F}(\vec{k},t) = 0$, so $\Box F(\vec{x},t) = 0$. ∎

Wave Equations for Curved Geometries

The D'Alembertian operator $\Box$ describes waves in flat, three dimensional space. Waves moving in other surfaces are described a more general wave operator:

$$\Box = \sum_{\mu,\nu} a(x,t)\partial_\mu \left(a_{\mu\nu}(x,t)\partial_\nu \right).$$

- μ and ν label the $n+1$ coordinates;
- ∂_ν is the partial derivative with respect to the coordinate ν.
- $a(x,t)$ and $a_{\mu\nu}(x,t)$ are given functions which obey
  1. $a(x,t) > 0$;
  2. $\forall\, x, t$, the matrix $a_{\mu\nu}(x,t)$ has n positive and one negative eigenvalues

The Flat Wave Equation Recovered

Example (D'Alembertian)

Let $a(x,y,z,t)=1$, $a_{tt}(x,y,z,t)=-1$, $a_{xx}=a_{yy}=a_{zz}=1$, and all other $a_{\mu\nu}=0$. Then

$$\Box = \sum_{\mu,\nu\in\{x,y,z,t\}} a(x,t)\partial_\mu \left(a_{\mu\nu}(x,t)\partial_\nu\right) = -\partial_t^2 + \triangle.$$

The Wave Equation For Waves on a Sphere

Recall that the unit sphere can be described by coordinates θ, φ related to Cartesian coordinates by

$$x = \sin\theta\cos\varphi \quad ; \quad y = \sin\theta\sin\varphi \quad ; \quad z = \cos\theta$$

Example (Spherical Waves)

Waves on a sphere described by $a_{tt}(\theta,\varphi,t) = -\sin\theta$, $a_{\theta\theta}(\theta,\varphi,t) = \sin\theta$, $a_{\varphi\varphi}(\theta,\varphi,t) = \csc\theta$, $a(\theta,\varphi,t) = \csc\theta$, and all other $a_{\mu\nu} = 0$. Equivalently

$$\Box_{S_2} = \sum_{\mu,\nu\in\{\theta,\varphi,t\}} a(x,t)\partial_\mu\left(a_{\mu\nu}(x,t)\partial_\nu\right) = -\partial_t^2 + \csc\theta\,\partial_\theta(\sin\theta\,\partial_\theta) + \csc^2\theta\,\partial_\varphi^2$$

Limitations of the Fourier Transform

Previous example illustrates two major problems:

1. $\widehat{\Box_{S_2} F}(\vec{k}, t) \neq \left(-\partial_t^2 - k^2\right) \hat{F}(\vec{k}, t)$ because the coefficients of the derivatives depend on the variables.
2. $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. What do we even mean by the Fourier transform?

What to Do?

Two possible solutions:

1. Modify our tool, i.e., find an improved version of the Fourier transform (microlocal analysis);
2. Find a new tool.

We will take door #2, in particular, using Spectral Theory.

The Basic Idea

Rewrite the wave equation as $\partial_t^2 F(\vec{x}, t) = -(-\triangle F(\vec{x}, t))$. If we blithely treat $-\triangle$ as a "constant", the solution is

$$F(\vec{x}, t) = \cos(\sqrt{-\triangle}t)f(\vec{x}) + \sin(\sqrt{-\triangle}t)(-\triangle)^{-1/2}g(\vec{x}).$$

The goal of spectral theory is to give sense to the above expression.

1. Need to find a "basis" in which the Laplacian is "diagonal"
2. identical in spirit to matrix algebra
3. sensible strategy because the Laplacian is a linear operator

(Most of) Linear Algebra in One Easy Slide

1. $\exists$ vectors v, which we represent by n-tuples of $\mathbb{R} \vee \mathbb{C}$ numbers.
2. $\exists$ matrices M, which take vectors and turn them into new vectors called Mv.
3. If $v \neq 0$ and Mv are scalar multiples, then v is an eigenvector.
4. The ratio $\dfrac{(Mv)_i}{v_i} =: \lambda$ (where $v_i \neq 0$) is the eigenvalue of v.

The Spectral Theorem (Easy Version 1)

Theorem

If S is a real, symmetric $n \times n$ matrix, then

1. *S has n linearly independent eigenvectors;*
2. *all the eigenvalues of S are real;*
3. *S is orthogonally diagonalizable,*

$$S = UDU^{-1},$$

where D is a diagonal matrix containing the eigenvalues of S, and U is an orthogonal matrix whose columns are the corresponding orthogonalized unit eigenvectors of S.

Why We Care

If we want to compute S^k, then

$$S^k = \left(UDU^{-1}\right)^k = UD^kU^{-1}.$$

Indeed, for any function f:

$$f(S) = Uf(D)U^{-1}.$$

Why We Care

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Indeed, for any function f:

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Proof: for f analytic,

$$f(S) = \sum_{k=0}^{\infty} a_k (UDU^{-1})^k = \sum_{k=0}^{\infty} U(a_k D^k)U^{-1} = Uf(D)U^{-1}.$$

For f continuous/Borel, take the limit whatsie whatsie QED.

Reformulating the Spectral Theorem

Suppose we apply S to some vector v. Then

$$Sv = \sum_{\lambda} \lambda P_\lambda(v), \text{ with } P_\lambda(v) := (v \cdot v_\lambda) v_\lambda$$

The operators P_λ are called the projection operators of S.
To show use

1. the rules of matrix multiplication,
2. that columns/rows of U/U^{-1} are eigenvectors of S, and
3. that the diagonal of D consists of corresponding eigenvalues.

Spectral Theorem, Easy Version 2

Theorem (Real Spectral Theorem)

If S is a real, symmetric $n \times n$ matrix, the following identity holds:

$$S \to \sum_{\lambda \in \sigma} \lambda P_\lambda,$$

where σ is the spectrum (the collection of eigenvalues) of S and P_λ is the projection operator onto the eigenspace of λ. Moreover, $\sigma \subseteq \mathbb{R}$.

Corollary

$$\textit{For any } f : \mathbb{R} \to \mathbb{R}, \textit{ we can define } f(S) := \sum_{\lambda \in \sigma} f(\lambda) P_\lambda.$$

Adjoints and Friends

Definition (Adjoint Matrix)

Let M be a complex $n \times n$ matrix. The adjoint matrix M^* is given by $\bar{M}^T$.

Definition (Hermitian Matrix)

obeys $H^* = H$.

Definition (Unitary Matrix)

obeys $U^* = U^{-1}$.

The Spectral Theorem, First Generalization

Theorem (Complex Spectral Theorem)

If H is a Hermitian matrix, then the following identity holds:

$$H = UDU^{-1} \to \sum_{\lambda \in \sigma} \lambda P_\lambda,$$

with $\sigma \subseteq \mathbb{R}$ the spectrum of H, D a diagonal matrix containing the eigenvalues of H, U a unitary matrix of unit eigenvectors of H, and P_λ the projection operator onto the eigenspace of λ.

Corollary

$$\text{For any } f : \mathbb{R} \to \mathbb{C}, \; f(H) := \sum_{\lambda \in \sigma} f(\lambda) P_\lambda = Uf(D)U^{-1}.$$

Going to Infinite Dimensions

Although the Laplacian is similar to a matrix because it is linear, it differs as well because it is (in a sense to be explained below) an $\infty \times \infty$ matrix. We are thus multiplying and adding infinite rows of numbers and have to worry about limits. In order to give us sufficient control over these limits, we need to introduce the concept of Hilbert space.

Inner Product Spaces

Definition (Inner Product Space)

A complex vector space V and a form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ which is

1. positive definite:
   - A. $\langle v, v \rangle > 0 \, \forall \, v \neq 0$,
   - B. $\langle 0, 0 \rangle = 0$.
2. sesquilinear:
   - A. $\langle v, \alpha u + w \rangle = \alpha \langle v, u \rangle + \langle v, w \rangle \; \forall u, v, w \in V$ and $\alpha \in \mathbb{C}$.
   - B. $\langle \alpha u + w, v \rangle = \bar{\alpha} \langle u, v \rangle + \langle w, v \rangle \; \forall u, v, w \in V$ and $\alpha \in \mathbb{C}$.
3. (conjugate/Hermitian) symmetric: $\langle u, v \rangle = \overline{\langle v, u \rangle}$.

Notice that $((2A) \wedge (3)) \Rightarrow (2B)$ and $((2A) \wedge (2B)) \Rightarrow (1B)$.
Note: for V over $\mathbb{R}$, (2) $\to$ bilinearity and (3) $\to$ symmetry.

Hilbert Spaces

Definition

The standard metric on an inner product space is given by

$$d(u, v) = \sqrt{\langle u - v, u - v\rangle}.$$

Definition (Hilbert space $\mathcal{H}$)

An inner product space which is complete (as a metric space) in the standard metric $d(\cdot, \cdot)$.

First Example: $\mathbb{R}^n$

Example ($\mathbb{R}^n$)

Let $V = \mathbb{R}^n$ and let $\langle v, w \rangle = v \cdot w$, so $d(u, v) = \sqrt{(u - v) \cdot (u - v)} = \|u - v\|$. We know that $\mathbb{R}^n$ is complete in this metric, so it is a Hilbert space.

Another Finite Dimensional Example: $\mathbb{C}^n$

Non-Example

Let $V = \mathbb{C}^n$ and let $\langle v, w \rangle = v \cdot w$. Then $\langle v, v \rangle$ is not necessarily positive $\Rightarrow$ not an inner product space.

Example ($\mathbb{C}^n$)

Let $V = \mathbb{C}^n$ and let $\langle v, w \rangle = \bar{v} \cdot w$, so that $d(x, y) = \sqrt{(\bar{x} - \bar{y}) \cdot (x - y)} = \|x - y\|$. $\mathbb{C}^n$ is complete in this metric (it is simply the distance in $\mathbb{R}^{2n}$), so it is a Hilbert space.

An Infinite Dimensional Example: L^2 Spaces

Example ($L^2(\mathbb{R})$)

Let V be the space of all $\mathbb{C}$-valued functions f on $\mathbb{R}$ which obey

$$\int_{\mathbb{R}} |f|^2 dx < \infty.$$

The following inner product is well-defined and positive definite:

$$\langle f, g \rangle = \int_{\mathbb{R}} \bar{f} g \, dx.$$

The distance between two functions f and g is given by

$$d(f, g) = \sqrt{\int_{\mathbb{R}} |f - g|^2 dx}.$$

This space, called $L^2(\mathbb{R})$, is complete in this metric and is therefore a Hilbert space. It is infinite dimensional because there infinitely many linearly independent, mutually orthogonal functions in it.

Operators

Definition (Operators)

An operator O on a Hilbert space $\mathcal{H}$ is a linear map $\mathcal{H} \to \mathcal{H}$.

Example (Matrices)

An $n \times n$ matrix M gives rise to an operator on $\mathcal{H} = \mathbb{C}^n$ via matrix multiplication: $v \to Mv$.

Example (Laplacian)

Consider the functions $f \in L^2(\mathbb{R}^3)$ with square-integrable first and second derivatives. The Laplacian $\triangle$ is a linear operator on L^2 because $\triangle f$ is still square integrable (and so $\triangle$ maps $L^2 \to L^2$) and $\triangle af = a\triangle f$ for any constant a.

Adjoints

Definition (Adjoint Operator)

The adjoint O^* of an operator O on a Hilbert space $\mathcal{H}$ is the unique operator which obeys $\langle O^* v, w \rangle = \langle v, Ow \rangle \ \forall \ v, w \in \mathcal{H}$.

Example (Transpose Matrix)

Let M be a matrix operator on $\mathbb{R}^n$. Then $M^* = M^T$. Proof:

$$\left\langle M^T v, w \right\rangle = (M^T v) \cdot w = w^T (M^T v) = (Mw)^T v = v \cdot (Mw) = \langle v, Mw \rangle$$

Example (Adjoint Matrix)

For a complex matrix M acting on $\mathbb{C}^n$, must complex conjugate M, so $M^* = \bar{M}^T \Rightarrow$ adjoint operator coincides with adjoint matrix!

Self-Adjoint Operators

Definition (Self-Adjoint)

A self-adjoint operator obeys $H^* = H$.

Example (Hermitan Matrix)

Any Hermitian matrix M is clearly a self-adjoint operator.

Example (Laplacian)

Consider the Laplacian as an operator on $L^2(\mathbb{R}^3)$. For any two functions f and g in Dom $\triangle$ we have

$$\langle f, \triangle g\rangle = \int_{\mathbb{R}^3} \bar{f} \triangle g d^3x = -\int_{\mathbb{R}^3} \vec{\nabla}\bar{f} \cdot \vec{\nabla} g d^3x = \int_{\mathbb{R}^3} \triangle \bar{f} g d^3x = \langle \triangle f, g\rangle$$

Thus, the Laplacian is a self-adjoint operator on $L^2(\mathbb{R}^3)$.

The Spectral Theorem (Second Generalization)

Theorem (Generalized Spectral Theorem)

Let O be a self-adjoint operator a Hilbert space $\mathcal{H}$. Then the following identity holds:

$$O = \sum_{\lambda \in \sigma} \lambda P_\lambda.$$

where σ is the spectrum of O and P_λ is the projection operator onto the eigenspace of λ. Further, $\sigma \subseteq \mathbb{R}$.

Corollary

For any self-adjoint operator we have

$$f(O) = \sum_{\lambda} f(\lambda) P_\lambda.$$

Diagonalizing the Laplacian:

Notice that

$$\begin{aligned} \triangle f(\vec{x}) &= \left(-k^2 \hat{f}\right)^{\check{}}(\vec{x}) \\ &= \int_{\mathbb{R}^3} d^3k(-k^2)\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3y \frac{e^{-i\vec{k}\cdot\vec{y}}}{(2\pi)^{3/2}} f(y) \\ \text{"="} &\quad \int_{\mathbb{R}^3} d^3k(-k^2)\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} \left\langle \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}, f \right\rangle_{L^2(\mathbb{R}^3)} \end{aligned}$$

As $\triangle e^{i\vec{k}\cdot\vec{x}} = -k^2 e^{i\vec{k}\cdot\vec{x}}$, last formula looks like the GST, with

$$\sum_{\lambda} \to \int_{\mathbb{R}^3} d^3k, \qquad \lambda \to -k^2, \qquad P_{\vec{k}} f \to \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} \hat{f}(\vec{k})$$

Solving the Wave Equation Using Spectral Theory

Recall that our goal for going into spectral theory was to define the following expression:

$$F(\vec{x},t) = \cos(\sqrt{-\triangle}t) f(\vec{x}) + \sin(\sqrt{-\triangle}t)\,(-\triangle)^{-1/2} g(\vec{x}).$$

By the corollary to the GST:

$$\begin{aligned} F(\vec{x},t) &= \int_{\mathbb{R}^3} d^3k \left(\cos\left(\sqrt{-(-k^2)}t\right) P_{\vec{k}} f + \frac{\sin(\sqrt{-(-k^2)}t)}{\sqrt{-(-k^2)}} P_{\vec{k}} g \right) \\ &= \int_{\mathbb{R}^3} d^3k \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} \left(\cos(\omega t)\hat{f}(\vec{k}) + \frac{\sin(\omega t)}{\omega}\hat{g}(\vec{k}) \right) \\ &= \left(\hat{f}(\vec{k}) \cos \omega t + \hat{g}(\vec{k}) \frac{\sin \omega t}{\omega} \right)^{\vee}. \end{aligned}$$

Our two methods of solution agree!

Diagonalizing the Laplacian, a Second Look

At least schematically, the spectral theorem says that a self-adjoint operator can be can be "diagonalized" $H = UDU^{-1}$. In Fourier space, we have that

$$\widehat{\triangle f}(\vec{k}) = -k^2 \hat{f}(\vec{k}).$$

Thus, in "Fourier space" the "matrix" of the Laplacian is diagonal!

Theorem (Parseval's Theorem)

The Fourier transform is the unitary transformation which "diagonalizes" the Laplacian operator $\triangle$ on $L^2(\mathbb{R}^3)$, and the "diagonal operator" D is just multiplication by $-k^2$.

Solving the Wave Equation Using Spectral Theory, II

Schematically,

$$F(\vec{x},t) \text{“}=\text{''} U\cos(\sqrt{-D}t)U^{-1}f(\vec{x}) + U\sin(\sqrt{-D}t)(-D)^{-1/2}U^{-1}g(\vec{x}).$$

Using $U^{-1} \to \hat{}$, $D \to -k^2$, and $U \to \check{}$,

$$F(\vec{x},t) = \left(\hat{f}(\vec{k})\cos\omega t + \hat{g}(\vec{k})\frac{\sin\omega t}{\omega}\right)^{\vee},$$

as above.

The Solution for Waves on a Sphere

Recall our example equation on the sphere:

$$\Box_{S^2} = -\partial_t^2 + \csc\theta\partial_\theta(\sin\theta\partial_\theta) + \csc^2\theta\partial_\varphi^2 = -\partial_t^2 + \triangle_{S^2}.$$

Theorem

The solution to the Cauchy problem on the sphere is given by

$$F(\theta,\phi,t) = \cos(\sqrt{-\triangle_{S^2}}t)f(\theta,\phi) + \sin(\sqrt{-\triangle_{S^2}}t)(-\triangle_{S^2})^{-1/2}g(\theta,\phi)$$

for any initial values $f \in L^2(S^2)$ and $g \in L^2(S^2)$.

The Return of the General Wave Operator

Recall that a general wave operator has the form

$$\square = \sum_{\mu,\nu} a(x,t)\partial_\mu \left(a_{\mu\nu}(x,t)\partial_\nu \right).$$

where $a_{\mu\nu}$ is a real $(n+1) \times (n+1)$ matrix which has n positive and one negative eigenvalues. Wave equations of this sort describe the propagation of fundamental particles (like photons and electrons) in curved spacetime (i.e., a solution of general relativity). Since we observe photons in the world around us, a spacetime in which this operator has no solutions is physically unreasonable.

What I've Done

Theorem

(Seggev, 2004) Consider a 4-dimensional spacetime in which the coefficients of wave equation obey

1. $\partial_t a(\vec{x}, t) = 0$ *and* $\partial_t a_{\mu\nu}(\vec{x}, t) = 0 \; \forall \; \mu, \nu$*;*
2. *A mild "geometrical" condition.*

Then the wave equation can be recast in the form

$$\partial_t F(\vec{x}, t) = -ihF(\vec{x}, t).$$

Furthermore, h is self-adjoint on an appropriate Hilbert space, so the Cauchy problem has the solution

$$F(\vec{x}, t) = e^{-iht} f(\vec{x}).$$

Conclusions

1. The spectral theorem is a powerful tool for analyzing a large number of partial differential equations.
2. Using the spectral theorem, I have proven that large class of spacetimes possesses solutions to the wave equation, an important physical test of those spacetimes.
3. The Fourier transform is a powerful tool for analyzing PDEs with constant coefficients because it diagonalizes them in "Fourier space."
4. Mathematical physics is a Naco, not an Anti-Naco.