Dynamics in Stationary, Non-Globally Hyperbolic Spacetimes

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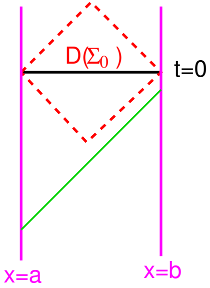
The Bottom Line

- There always are local solutions to the Klein-Gordon equation.
- In globally hyperbolic spacetimes, there exist global solutions with a host of important properties (below).
- The present work establishes the existence of global solutions to the wave equation in causal, stationary, non-globally hyperbolic spacetimes.
- Further, there is a prescription for assigning solutions to initial data which preserves important properties of the well-posed problem.

(Non-)Global Hyperbolicity

- The domain of dependence $D(\Sigma_0)$ is the set of points p such that every inextendible timelike curve through p intersects Σ_0.
- Globally hyperbolic spacetimes M have a Cauchy surface Σ_0 (for which $D(\Sigma_0) = M$).



Well-posedness

- Global-hyperbolicity guarantees the well-posedness of initial value problem for scalar test fields:
  - there is a unique solution throughout spacetime for given initial data,
  - solutions depend continuously on initial data, and
  - smooth initial data produce smooth solutions.
- In stationary spacetimes, solutions also conserve energy.

Non-Globally-Hyperbolic Spacetimes

- In general, non-globally hyperbolic spacetimes have an ill-posed initial value problem.
- Wald (1980) and Wald and Ishibashi (2003) treated the case of static spacetimes in complete generality.
- The present work shows that a prescription exists in a large class of general stationary (not necessarily static) spacetimes.

Stationary Spacetimes

- (M, g_{ab}) is stationary if it has an everywhere timelike, complete Killing vector field t^a.
- Black hole solutions are not stationary.
- A static spacetime has time-reversal symmetry as well.

Plan of Attack

The general plan is as follows:

1. Construct a suitable Hilbert space of initial data.
2. Convert the PDE problem into a Hilbert space problem.
3. Solve the Hilbert space problem.
4. Convert back and show that the result is a sensible PDE solution.

The Hilbert Space

The energy Hilbert space $\mathcal{H}_{\mathcal{A}}$ is the completion of $C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$ in the inner product

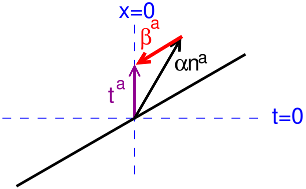
$$\langle \Phi \mid \Phi \rangle := \int_\Sigma d\gamma T_{ab} n^a t^b .$$

Lapse and Shift

Recall that the lapse function α and shift-vector β^a are defined by

$$t^a = \alpha n^a + \beta^a,$$

where $\beta^a n_a = 0$. Note that $-t^a t_a = \alpha^2 - \beta^2$.



The Klein-Gordon Equation

- The Klein-Gordon equation is a second order hyperbolic differential equation:

$$(\nabla^a \nabla_a - m^2)\varphi = 0.$$

- Using the canonical momentum $\pi = n^a \nabla_a \varphi$, and letting $\Phi = (\varphi, \pi)$, this equation may be rewritten as a first order system:

$$\frac{\partial}{\partial t}\Phi = -h\Phi.$$

Properties of h

- h's explicit form is

$$-h = \begin{bmatrix} \beta^a D_a & \alpha\pi \\ D^a(\alpha D_a) - \alpha m^2 & -(D_a\beta^a) - \beta^a D_a \end{bmatrix}$$

- h is a 2×2 matrix operator containing only spatial derivatives.
- The form of h depends on the choice of slicing.
- h, acting on $C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$, is anti-Hermitian in the energy inner product.

Assumptions

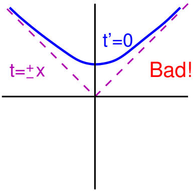
- Restrict attention to fields with

$$m^2 > 0. \qquad \text{(PosMass)}$$

- It is necessary that the slicing obey

$$\alpha - \frac{\beta_a \beta^a}{\alpha} \geq \epsilon > 0. \qquad \text{(NonNull)}$$

This implies that $\alpha \geq \epsilon$ and $\alpha^2 - \beta^2 \geq \epsilon^2$.



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Recall that $$\frac{\partial}{\partial t}\Phi(t,x) = -h\Phi(t,x).$$

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$$\text{Recall that} \quad \frac{\partial}{\partial t}\Phi(t,x) = -h\Phi(t,x).$$

4. Choose a skew-adjoint extension h^{SA} of h and use the spectral theorem to define

$$\Phi_t(x) = e^{-h^{SA}t}\Phi_0(x).$$

Existence of Extension

Theorem I

Let (M, g_{ab}) be a stationary spacetime, and consider a minimally coupled Klein-Gordon equation subject to (PosMass). If (Σ_t, γ_{ab}) is a foliation of satisfying (NonNull), then h possesses at least one skew-adjoint extension. Further, this extension h^I is invertible.

Properties of Solutions

Theorem II

Assume the conditions of Theorem I hold. Let Φ_0 be smooth data of compact support. If Φ is the solution defined via the prescription for any h^{SA} and Ψ the maximal Cauchy evolution of Φ_0, then

(a) $\Phi = \Psi$ *within* $D(\Sigma_0)$,

(b) Φ *varies continuously with initial data,*

(c) smooth data of compact support give rise to smooth solutions, and

(d) Φ *conserves energy.*

The Static Case

Theorem III

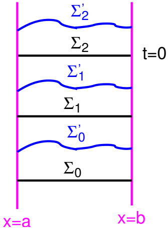
Let (M, g_{ab}) be a static spacetime obeying (NonNull) in the static slicing. If (PosMass) holds, then h is essentially skew-adjoint. Further, the stationary spacetime prescription agrees with a definite prescription in the Wald-Ishibashi formalism for static spacetimes.

Conclusions

- A non-empty class of prescriptions for defining dynamics can be given in stationary spacetimes obeying the mild condition (NonNull).
- Any prescription in this class automatically conserves energy.
- In the static case, there is only one prescription in the class. It corresponds to a definite prescription in Wald's formalism.
- As an added bonus, linear field quantization is possible.

Open Questions

- Is the extension h^l unique?
- How do the classes in different slicings compare?



- In the static case, this formalism can be modified to include all Wald-Ishibashi dynamics. Is something similar true in the general case?