# Dynamics in Stationary, Non-Globally Hyperbolic Spacetimes

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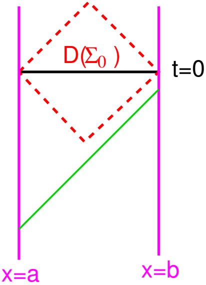
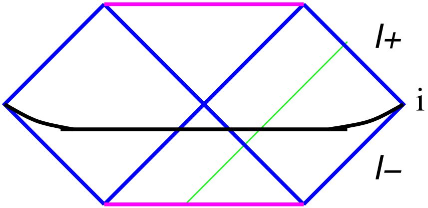
# (Non-)Global Hyperbolicity








- The domain of dependence $D(\Sigma_0)$ is the set of points *p* such that every inextendible timelike curve through *p* intersects $\Sigma_0$.
- Globally hyperbolic spacetimes $M$ have a Cauchy surface $\Sigma_0$ for which $D(\Sigma_0) = M$.



# Well-posedness

- Global-hyperbolicity guarantees the well-posedness of initial value problem for scalar test fields:
  - There is a unique solution throughout spacetime for given initial data, and
  - solutions depend continuously on initial data.
- In general, non-globally hyperbolic spacetimes have an ill-posed initial value problem.

# The Goal

- There always are local solutions to the wave equation. The present work is concerned with global solutions.
- It is desirable to find a space of solutions which preserve important properties of the well-posed problem, i.e., find a prescription for assigning solutions to initial data.
- There may be more than one such space.

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# Properties to Preserve

The prescription for defining dynamics in non-globally hyperbolic spacetimes should do the following:

- solve the wave equation,
- agree with the PDE solution within the domain of dependence,
- conserve energy,
- be smooth when initial data is smooth and of compact support, and
- depend continuously on initial data.

# Plan of Attack

The general plan is as follows:

1. Construct a suitable Hilbert space of initial data.
2. Convert the PDE problem into some Hilbert space problem.
3. Solve the Hilbert space problem.
4. Convert back and show that the result is a sensible PDE solution.

# Selected History

- Wald (1980) studied Klein-Gordon fields on static spacetimes. He gave a class of prescriptions using the “spatial part” of the KG equation as an operator on an appropriate Hilbert space of square integrable initial data.
- Wald and Ishibashi (2003) proved that any “reasonable” prescription is in the class proposed by Wald.
- This work gives a similar class of prescriptions for general stationary spacetimes but using an energy Hilbert space.

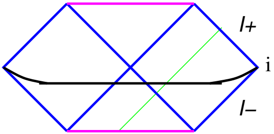
# Applications of Prescriptions

Wald's prescription has been used in at least four different lines of research:

- field quantization
- stability of naked singularities
- definition of "quantum singularity"
- AdS/CFT.

# Stability of Naked Singularities

- The Schwarzschild solution is stable against linear perturbation in its initial data.



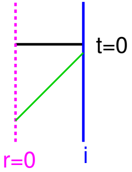

- Naked singularities give rise to ill-posed initial value problems. What does it mean to perturb the singularity?
- Stalker has proven a “Stichartz” (decay) estimate for spherically symmetric scalars evolving according to one Wald prescription on a super-extremal Reissner-Nordström background.
- This estimate establishes mode-by-mode stability.

# Definition of "quantum singularity"

- Geodesic incompleteness is the generally accepted definition of a singularity in classical GR.
- Horowitz and Marolf proposed that a static spacetime be quantum mechanically non-singular if the class of Wald prescriptions contains only one member.
- Some classically singular spacetimes are quantum mechanically non-singular (e.g., dilatonic black holes), but some classically non-singular spacetimes become quantum mechanically singular (e.g., AdS).

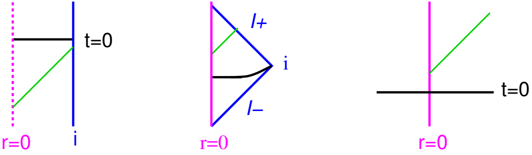
# AdS/CFT

- AdS/CFT relates the behaviour of a "boundary" conformal field theory to the behaviour of a "bulk" theory living on all of AdS.
- There is more than one possible bulk theory!
- Wald and Ishibashi have explicitly characterized all the dynamics in the Wald class for AdS.



# Stationary Spacetimes

- $(M, g_{ab})$ is stationary if it has a Killing vector field $t^a$ which is everywhere timelike.
- Black hole solutions are not considered stationary.
- Attention will be restricted to stably causal spacetimes.
- Examples
  AdS, super-extremal Reissner-Nordström, cosmic strings.




- Non-examples: sub-extremal Reissner-Nordström, “unlifted” AdS.

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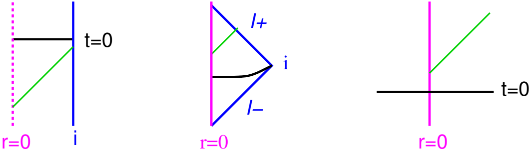


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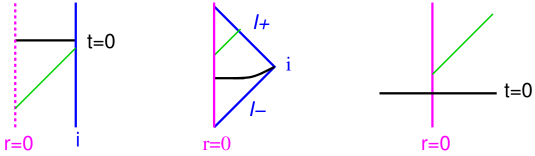


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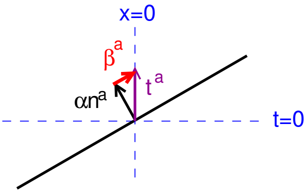
# Lapse & Shift

- Let $(\Sigma_0, \gamma_{ab})$ be a spatial slice. The projection of $t^a$ onto the unit normal $n^a$ to $\Sigma_0$ is called the lapse function $\alpha$, and this defines the shift-vector $\beta^a$ via

$$t^a = \alpha n^a + \beta^a.$$




- Note that $\beta^a n_a = 0$ and $-t^a t_a = \alpha^2 - \beta^2$.
- If there exists a slicing in which $\beta^a = 0$, then $(M, g_{ab})$ is called static.

# The Klein-Gordon Equation

- The Klein-Gordon equation is a second order hyperbolic differential equation:

$$(\nabla^a \nabla_a + m^2)\varphi = 0.$$

- Using the canonical momentum $\pi = n^a \nabla_a \varphi$, it may be rewritten as a first order system:

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \beta^a D_a \varphi + \alpha\pi \\ \frac{\partial \pi}{\partial t} &= \left[ D^a(\alpha D_a) - \alpha m^2 \right] \varphi - \left[ (D_a \beta^a) + \beta^a D_a \right] \pi \end{aligned}$$

In the above, $D_a$ is the covariant derivative of $(\Sigma_0, \gamma_{ab})$.

- Let $\Phi = (\varphi, \pi)$ and rewrite the above as

$$\frac{\partial}{\partial t}\Phi = -h\Phi.$$

# The Klein-Gordon Equation: Static Spacetimes

- In the static slicing, the above becomes

$$\frac{\partial \varphi}{\partial t} = \alpha\pi$$
$$\frac{\partial \pi}{\partial t} = \left[D^a(\alpha D_a) - \alpha m^2\right]\varphi$$

- These can be combined to

$$\frac{\partial^2}{\partial t^2}\varphi = -S\varphi, \quad S = -\alpha D^a(\alpha D_a) + \alpha^2 m^2.$$

- $S$ is Hermitian w.r.t. the volume element $\alpha^{-1}d\gamma$:

$$\int_\Sigma \alpha^{-1}d\gamma\, \varphi(S\psi) = \int_\Sigma \alpha^{-1}d\gamma\, (S\varphi)\psi.$$

# The Hilbert Space

The classical Hamiltonian in this slicing is

$$H(\Phi) = \frac{1}{2}\int_\Sigma d\gamma \Phi^T \mathcal{A}\Phi, \quad \mathcal{A} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} h$$

Note that this is equal to $\int_\Sigma d\gamma T_{ab} n^a t^b$. The energy Hilbert space $\mathcal{H}_\mathcal{A}$ is the completion of $C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$ in the inner product

$$\langle \Phi \mid \Psi \rangle := \int_\Sigma d\gamma \Phi^T \mathcal{A}\Psi.$$

$h$ is anti-Hermitian in the energy inner product.

# Assumptions

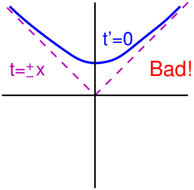
- To ensure that all the elments of $\mathcal{H}_{\mathcal{A}}$ are functions, assume that

$$m^2 > 0. \qquad \text{(PosMass)}$$

- Also assume that

$$\alpha - \frac{\beta_a \beta^a}{\alpha} \geq \epsilon > 0. \qquad \text{(NonNull)}$$

This implies that $\alpha \geq \epsilon$ and $\alpha^2 - \beta^2 \geq \epsilon^2$.



# The Symplectic Form

The two assumptions (PosMass) and (NonNull) ensure that the symplectic form

$$\sigma\left[(\phi_1,\pi_1),(\phi_2,\pi_2)\right] = \int d\gamma(\phi_1\pi_2 - \phi_2\pi_1)$$

is continuous on $\mathcal{H}_{\mathcal{A}}$. This continuity will be crucial in the coming analysis.

# The Prescription(s)

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4. Therefore, choose a skew-adjoint extension $h^{SA}$ of $h$ and use the spectral theorem to define
   $$\Phi_t(x) = e^{-h^{SA}t}\Phi_0(x).$$

# The Definition of an Operator

- A linear operator $A$ on $\mathcal{H}$ is a map from a dense vector subspace of $\mathcal{H}$, called Dom $A$, to $\mathcal{H}$.
- If Dom $C \supseteq$ Dom $A$ and $C\psi = A\psi$ $\forall$ $\psi \in$ Dom $A$, then $C$ is an extension of $A$, denoted $C \supseteq A$.
- For a bounded operator $B$, there is always a unique continuous extension to all of $\mathcal{H}$. It is called $\bar{B}$, the closure of $B$.
- For an unbounded operator, taking the closure will produce an operator which is not defined on all of $\mathcal{H}$.
- Indeed, an unbounded anti-Hermitian operator can never be defined on all of $\mathcal{H}$.

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# Adjoints *et cetera*

- Definition
  Dom $A^*$ contains $u \in \mathcal{H}$ such that $\exists v \in \mathcal{H}$ with $\langle u \mid A\psi \rangle = \langle v \mid \psi \rangle \ \forall\, \psi \in$ Dom $A$, and $A^* u = v$.
- An operator is (anti-)Hermitian if $\langle A\Phi \mid \Psi \rangle = (-) \langle \Phi \mid A\Psi \rangle \ \forall\, \Phi, \Psi \in$ Dom $A$. For these, Dom $A \subseteq$ Dom $A^*$.
- A self-adjoint (skew-adjoint) operator is an (anti-)Hermitian operator such that Dom $A =$ Dom $A^*$.
- If $C \subseteq A$, then $A^* \supseteq C^*$. Does an (anti-)Hemitian operator always have a self-adjoint (skew-adjoint) extension?

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- If $C \subseteq A$, then $A^* \supseteq C^*$. Does an (anti-)Hemitian operator always have a self-adjoint (skew-adjoint) extension? **No!**

# von Neumann's Theorem

- Let $A$ be a Hermitian operator an a complex Hilbert space, and let $n_\pm = \text{Dim Ker}(A^* \mp i)$.
- von Neumann's Theorem says $A$ has self-adjoint extensions iff $n_+ = n_-$, in which case there is a $U(n_+)$ family of extensions.
- If $n_+ = n_- = 0$, $A$ is called essentially self-adjoint.
- If $A$ is an operator on a real Hilbert space, look at self-adjoint extensions of the complexified operator with domain invariant under complex conjugation.
- For a skew-adjoint operator $A$, look at $iA$.

# The Spectral Theorem

- The spectral theorem applies to both self- and skew-adjoint operators.
- The key idea of the spectral theorem is that the operator can be represented as a sum (or integral) over orthogonal projectors:

$$A = \sum_{\lambda \in \text{Spec}} \lambda P_\lambda .$$

- For a skew-adjoint operator, the spectrum is purely imaginary. Hence

$$e^A = \sum_{\lambda \in i\mathbb{R}} e^\lambda P_\lambda$$

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# $h$ Revisited

- $h$, as a differential operator, is only anti-symmetric on $C_0^\infty$ data.
- Dom $h^*$ will aways be larger because it contains less differentiable functions in its domain.
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- Dom $h^*$ will aways be larger because it contains less differentiable functions in its domain.
- The key question is what are $n_+$ and $n_-$? I don't know.
- I explicitly found one extension without computing the indices.

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1. Choose a slicing obeying (NonNull) and construct the space $\mathcal{H}_{\mathcal{A}}$.
2. Define the operator $h$ as above.

   Recall that $$\frac{\partial}{\partial t}\Phi(t,x) = -h\Phi(t,x).$$

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Notice that $\Phi_t$ is defined at every point of space, and the transformation from $\Phi_0$ to $\Phi_t$ is unitary.

# Existence of Extension

## Theorem I

*Let $(M, g_{ab})$ be a stationary spacetime, and consider a minimally coupled Klein-Gordon equation subject to (PosMass). If $(\Sigma_t, \gamma_{ab})$ is a foliation of satisfying (NonNull), then h possesses at least one skew-adjoint extension . Further, this extension $h^I$ is invertible and preserves the symplectic form.*

**Outline of Proof:** Since $\sigma$ is continuous and skew-symmetric, it has an associated skew-adjoint operator $T$. $T^{-1}$ can be shown to exist as a skew-adjoint operator, which is also an extension of $h$.

# Properties of Solutions





## Theorem II

*Assume the conditions of Theorem I hold. Let $\Phi_0$ be smooth data of compact support, $\Phi_t$ the family of vectors defined via the prescription, and $\Psi$ the maximal Cauchy evolution of $\Phi_0$. If $\Phi(p, t) = \Phi_t(p)$, then $\Phi = \Psi$ within the domain of dependence $D(\Sigma_0)$. Also, smooth data of compact support give rise to smooth solutions.*

**Idea of Proof:** The failure of the solutions to agree within the domain of dependence would violate local conservation of the symplectic form. Elliptic regularity shows that $\Phi$ is smooth on each fixed slice. Together, these show that $\Phi$ is smooth.

More details on the proof.

# The Static Case

## Theorem III

*Let $(M, g_{ab})$ be a static spacetime obeying (NonNull) in the static slicing. If (PosMass) holds, then h is essentially skew-adjoint. Further, the first order prescription agrees with Wald's prescription:*

$$\varphi_t = \cos\left(S_F^{\frac{1}{2}} t\right)\varphi_0 + S_F^{-\frac{1}{2}} \sin\left(S_F^{\frac{1}{2}} t\right)\alpha\pi_0$$

*where $S_F$ is the Friedrichs extension of $S$, the spatial part of the Klein-Gordon equation, regarded as on operator on $L^2(\alpha^{-1} d\gamma)$.*

**Ingredients of proof:** Positivity of energy and lots of calculation.

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More on the Friedrichs extension.

# Field Quantization: The Past

- Kay showed that in a globally hyperbolic spacetime, $h$ is essentially skew-adjoint.
- That proof depended on the well-posedness of the initial value problem.
- Further, $|h^I|^{-1}$ is an appropriate "complex structure" for field quantization. This complex structure provides the rigorous definition of the "frequency splitting" vacuum.
- A complex structure is the thing which tells you what are the creation operators and what are the annihilation operators.

# Field Quantization: The Present

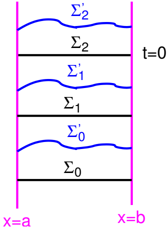
- In the present case, the dynamics are not well-defined *a priori* and $h$ need not be essentially skew-adjoint.
- However, it can be shown that $|h^I|^{-1}$ is still an appropriate complex structure for the quantum theory.
- First general result in non-globally hyperbolic spacetimes?

# Conclusions

- A non-empty class of prescriptions for defining dynamics can be given in stationary spacetimes obeying the mild condition (NonNull).
- Any prescription in this class automatically conserves energy.
- In the static case, there is only one prescription in the class. It corresponds to a definite prescription in Wald's formalism.
- As an added bonus, linear field quantization is possible.

# Open Questions

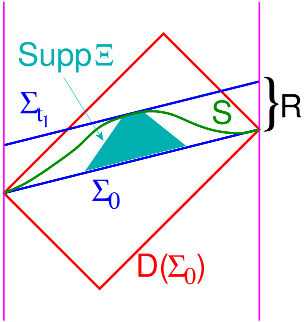
- Is $h$ essentially skew-adjoint?
- How do the classes in different slicings compare?




- In the static case, this formalism can be modified to include all “reasonable” dynamics. Is something similar true in the general case?

# Proof of Theorem II, Part 1

- Suppose $\Delta(t,\cdot) := \Psi(t,\cdot) - \Phi_t \neq 0$.




- Let $\Xi(t_1,\cdot)$ be a smooth function such that

$$\sigma\left(\Xi(t_1,\cdot),\Delta(t_1,\cdot)\right) \neq 0$$

- Extend $\Xi(t_1,\cdot)$ to a smooth solution of Klein-Gordon in the whole region $R$.

- But, $\Delta(0,\cdot) = 0$, so $\sigma\left(\Xi(0,\cdot),\Delta(0,\cdot)\right) = 0$
- Contradiction!
- Thus, $\Phi = \Psi$ within $D(\Sigma_0)$.

# Proof of Theorem II, Part 2

- Notice that $\mathcal{H}_{\mathcal{A}} \subseteq H^1_{\text{loc}} \oplus H^0_{\text{loc}}$.
- By Stone's Theorem

$$\Phi_0 \in \text{Dom}h^{SA} \Leftrightarrow \Phi_t \in \text{Dom}h^{SA} \Leftrightarrow h\Phi_t \in H^1_{\text{loc}} \oplus H^0_{\text{loc}}.$$

- Let $X(F) := -\left\langle h^{SA}F \,\middle|\, \Phi_t \right\rangle_{\mathcal{A}}$.
- From the explicit from of $A$, $X \in H^{-1}_{\text{loc}} \oplus H^{-1}_{\text{loc}}$.
- $X$ obeys the differential equation

$$X = Ah\Phi_t = \begin{bmatrix} \left(\alpha^2\gamma^{ab} + \beta^a\beta^b\right) D_a D_b \pi_t + (D_a\beta^a) D^b D_b \varphi_t + \text{l.o.t} \\ \left(\alpha^2\gamma^{ab} + \beta^a\beta^b\right) D_a D_b \varphi_t + \text{l.o.t} \end{bmatrix}.$$

- Thus, $\pi_t \in H^1_{\text{loc}}$ and $\varphi_t \in H^2_{\text{loc}}$. Induct.
- Since $\Phi_t = \Psi(t, \cdot)$ within $D(\Sigma)$, smoothness on each slice implies smoothness throughout spacetime.

Back to Theorem II

# Quadratic Forms

Let $A$ be a positive Hermitian operator on $\mathcal{H}$.

- Associated to $A$ is a quadratic form

$$Q(\phi, \psi) = \langle \phi \,|\, A\psi \rangle \,.$$

- The Friedrichs form domain, $\mathcal{Q}_F$, is the completion of Dom $A$ in the norm

$$\langle \phi \,|\, \phi \rangle_F = \langle \phi \,|\, \phi \rangle + \langle \phi \,|\, A\phi \rangle \,.$$

- Note that $\mathcal{Q}_F \subseteq \mathcal{H}$, and that $Q$ naturally extends to a larger quadratic form $Q_F$ on $\mathcal{Q}_F$.

# The Friedrichs Extension

Let $A$ be a positive Hermitian operator on $\mathcal{H}$.

- The Friedrichs extension $A_F$ of $A$ is the unique self-adjoint operator obeying

$$\text{Dom } A_F^{1/2} = \mathcal{Q}_F, \quad Q_F(\phi, \psi) = \left\langle A_F^{1/2}\phi \,\middle|\, A_F^{1/2}\psi \right\rangle.$$

- Any other positive self-adjoint extension $A_E$ of $A$ obeys

$$\text{Dom } A_E^{1/2} \supseteq \text{Dom } A_F^{1/2}.$$

- In this sense, the Friedrichs extension is the smallest positive self-adjoint extension of $A$.

# Other Dynamics: The Problem

Wald and Ishibashi showed that any reasonable prescription must be of the form

$$\varphi_t = \cos\left(S_E^{\frac{1}{2}} t\right) \varphi_0 + S_E^{-\frac{1}{2}} \sin\left(S_E^{\frac{1}{2}} t\right) \alpha\pi_0.$$

Theorem III says that *h* is essentially skew-adjoint. What happened to the other dynamics?

# Other Dynamics: The Solution





To reproduce the dynamics corresponding $S_E$:

1. Define $\tilde{\mathcal{H}}_{\mathcal{A}} := \text{Dom } S_E^{1/2} \oplus L^2(\alpha^{-1} d\gamma)$.
2. Define $\tilde{h}$ on $\tilde{\mathcal{H}}_{\mathcal{A}}$ using the spectral resolution of $S_E$ on $L^2(\alpha^{-1} d\gamma)$.
3. $\tilde{h}$ is already skew-adjoint, so the solution is

$$\Phi_t = e^{-\tilde{h}t}\Phi_0.$$

Back to Theorem III

# Definition of “Reasonable”

Wald and Ishibashi said a “reasonable” prescription for static spacetimes must:

- solve the wave equation,
- agree with the PDE solution within the domain of dependence,
- conserve energy,
- be smooth when initial data is smooth and of compact support,
- depend continuously on initial data,
- be time symmetric, and
- obey a certain limit condition.

Notice that the first 5 properties were required for the present prescription as well.

# Higher Spin Fields



Example

Maxwell's equations: $\nabla^a \nabla_a A_b - R_{ba} A^a = 0$.

- Components do not decouple, so cannot use scalar equations.
- The "vector energy" $\int_S d\gamma A^b \nabla^a \nabla_a A_b$ is not positive-definite.
- Spin $s > 1$ fields not locally well posed, in general.

Conclusion: a general prescription such as was given here would be difficult to achieve.

# Interacting Fields

## Example

A polynomial non-linearity $\nabla^a \nabla_a \varphi = P(\varphi)$.

- Probably the best that can be done is perturbation theory:

$$\nabla^a \nabla_a \varphi^{(n+1)} = s_n \left( P, \varphi^{(0)}, \ldots, \varphi^{(n)} \right).$$

- Need to solve the free equation with source:

$$\frac{\partial \Phi(p,t)}{\partial t} = -h\Phi(p,t) + S_n(p,t),\ S_n = \frac{1}{\alpha} \begin{pmatrix} 0 \\ s_n \end{pmatrix}.$$

- Solution: $\Phi_t^{\mathrm{in}} = e^{-ht} \int_0^t e^{h\tau} S_n(\tau) d\tau + e^{-ht} \Phi_0$.
- However, $S_n$ may not lie in $\mathcal{H}_{\mathcal{A}}$.
- For quantum theory, need to also do renormalization.