The Klein-Gordon Equation Meets the Cauchy Horizon

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Relativistic Wave Equations

At the present time, our best theory for describing nature is Quantum Field Theory. Each spin has a relativistic wave equation describing the propagation of free particles. For the lowest three spins, these are:

- 0: Klein-Gordon Equation
- 1/2: Dirac Equation (or Weyl Equations)
- 1: Maxwell's Equations

This talk will focus on the simplest of these, the Klein-Gordon equation, and describe some of the complications which arise in the classical theory when gravitational effects are turned on.

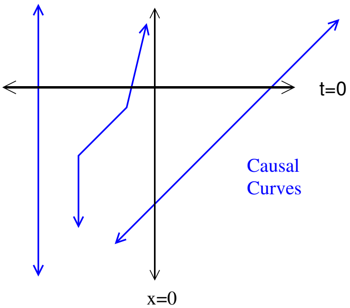
Causal Curves

Definition

A causal curve is a curve in spacetime which corresponds to an observer moving at or below the speed of light.

Causal curves:

- indicate possible information flow
- have slope $\frac{dt}{dx} \geq 1$.



Cauchy Horizons

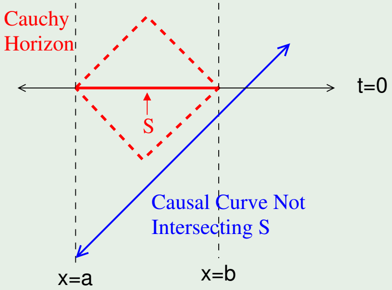
Definition

Given a spatial surface S, let A be the set of points from which you can follow a causal curve and still avoid hitting the surface S. The boundary of the set A is called the Cauchy horizon of S.

First Example: A Line Segment

Example

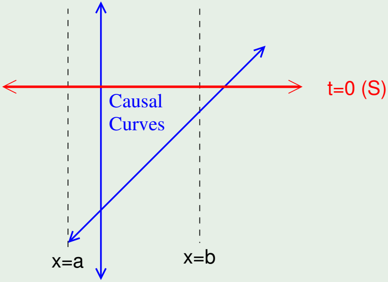
S is the line segment $a \leq x \leq b$ on the t-axis (y and z are ignored). The dashed diamond is the Cauchy horizon, and the area outside of it is the set A. This shown by drawing a causal curved which does not intersect S.



Second Example: $t = 0$

Non-Example

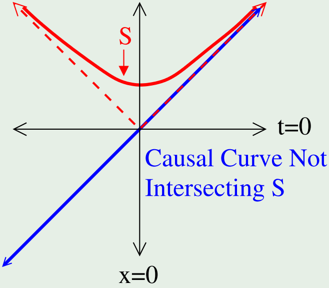
S is the whole $t = 0$ axis. There are no causal curves which do not intersect it, so the set A is empty. There is no Cauchy horizon.



Third Example: Hyperbola

Example

S is the hyperbola $t^2 - x^2 = 1$, $t > 0$. The null curves $t = |x|$ do not intersect it, and form the Cauchy horizon. The set A lies outside the "upper wedge".



The Klein-Gordon Equation

Definition

The flat space Klein-Gordon equation is given by

$$(\Box - m^2)\varphi = (-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 - m^2)\varphi$$

$$= \left(\sum_{\mu,\nu=0}^{3} \eta^{\mu\nu} \partial_\mu \partial_\nu - m^2 \right) \varphi = 0.$$

$\eta^{\mu\nu}$ is the matrix

$$\eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Initial Value Problem

Definition

The initial value problem or IVP is to find a function $\varphi(\vec{x}, t)$ which obeys

1. $(\Box - m^2)\varphi(\vec{x}, t) = 0$ in $\mathbb{R}^4$.
2. $\varphi(\vec{x}, 0) = \varphi_0(\vec{x})$ for $\vec{x} \in \mathbb{R}^3$.
3. $\dfrac{\partial \varphi}{\partial t}(\vec{x}, 0) = \pi_0(\vec{x})$ for $\vec{x} \in \mathbb{R}^3$.

Fourier Transformed Equation

Consider the Fourier-transform of φ

$$\hat{\varphi}(\vec{k}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \varphi(\vec{x}, t) e^{-i\vec{k}\cdot\vec{x}} d^3x.$$

The Klein-Gordon equation takes the form

$$\left(-\partial_t^2 - k^2 - m^2\right) \hat{\varphi}(\vec{k}, t) = 0.$$

For each k, this is a constant coefficient ODE in time, whose solutions are sines and cosines.

Solution of the Initial Value Problem

Theorem

The solution of the initial value problem is

$$\varphi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(\hat{\varphi}_0(\vec{k}) \cos \omega t + \hat{\pi}_0(\vec{k}) \frac{\sin \omega t}{\omega} \right) e^{i\vec{k}\cdot\vec{x}} d^3k$$

$$=: \mathfrak{F}^{-1} \left(\hat{\varphi}_0(\vec{k}) \cos \omega t + \hat{\pi}_0(\vec{k}) \frac{\sin \omega t}{\omega} \right),$$

where $\omega = \sqrt{k^2 + m^2}$.

Proof of the Theorem

By construction, φ obeys the equation, so we just need to check:

1. Does $\varphi(\vec{x}, t)$ have the right initial value? Yes:

$$\varphi(\vec{x}, 0) = \mathfrak{F}^{-1}\left(\hat{\varphi}_0(\vec{k})\cos(\omega \cdot 0) + \hat{\pi}_0(\vec{k})\frac{\sin(\omega \cdot 0)}{\omega}\right)$$

$$= \mathfrak{F}^{-1}\left(\hat{\varphi}_0(\vec{k}) \cdot 1 + 0\right) = \varphi_0(\vec{x}).$$

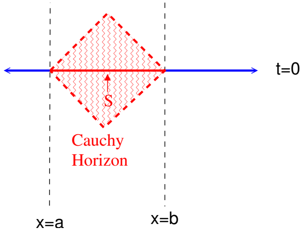
2. Does $\varphi(\vec{x}, t)$ have the right initial derivative? Yes:

$$\frac{\partial \varphi}{\partial t}(\vec{x}, 0) = \mathfrak{F}^{-1}\left(\hat{\varphi}_0(\vec{k})(-\omega \sin(\omega \cdot 0)) + \hat{\pi}_0(\vec{k})\frac{\omega \cos(\omega \cdot 0)}{\omega}\right)$$

$$= \mathfrak{F}^{-1}\left(0 + \hat{\pi}_0(\vec{k}) \cdot 1\right) = \pi_0(\vec{x}).$$

Initial Value Problem for the Strip

- Specifying initial data on $t = 0$ gave a unique solution.
- Suppose we only specify initial data on the strip $a \leq x \leq b$.



- Many global solutions restrict to the same local solution.
- Conversely, we can solve the IVP within the Cauchy horizon, but to find a global solution we must extend the solution across the horizon.

Cauchy Horizons and Global Solutions

- Specifying initial data on a spatial surface without a Cauchy horizon gives a global solution.
- Specifying initial data on a spatial surface with a Cauchy horizon gives a local solution.
- A local solution may extend to 0, 1, or many global solutions.

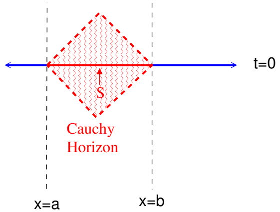
Cauchy Horizons and Global Solutions

- Specifying initial data on a spatial surface without a Cauchy horizon gives a global solution.
- Specifying initial data on a spatial surface with a Cauchy horizon gives a local solution.
- A local solution may extend to 0, 1, or many global solutions.
- So we should only solve the IVP for surfaces without a Cauchy horizon, right?

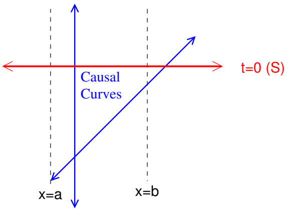
Non-Globally-Hyperbolic Spacetimes

Definition

A spacetime is called non-globally-hyperbolic if every spatial surface has a Cauchy horizon.



Non-Globally-Hyperbolic



"Ordinary" Case

Examples of Non-Globally-Hyperbolic Spacetimes

Example

Charged and/or rotating black holes.

Example

Anti-de Sitter Space.

Example

Cosmic string spacetimes.

Our Goal

- To solve the Klein-Gordon equation in non-globally-hyperbolic spacetimes, we must come up with a method for extending solutions past the Cauchy horizon.
- We will use an operator approach to do so.

Preview

- The Laplacian in spherical coordinates is

$$\nabla^2_{\mathbb{S}^2\times\mathbb{R}} = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial^2_\phi$$

- To understand where this expression comes from, we need to understand the meaning of the matrix $\eta^{\mu\nu}$ and its inverse $\eta_{\mu\nu}$.
- In this section, we will explore the meaning of the matrix $\eta_{\mu\nu}$.
- In the next, we will return to the Klein-Gordon equation.

Review of Special Relativity

Definition

Recall that in special relativity, the unique, observer independent quantity associated to two events is the spacetime interval

$$s^2 = I = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Definition

A (4-dimensional) displacement vector is called timelike, spacelike, or null depending on whether the interval I is negative, positive, or zero.

Special Relativity in Infinitesimal form

The spacetime interval can be written in infinitesimal form:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu$$

In this case, $\eta_{\mu\nu} = (\eta^{\mu\nu})^{-1} = \eta^{\mu\nu}$.

Definition

The matrix $\eta_{\mu\nu}$ is called the metric of flat spacetime in Cartesian coordinates.

Tangent Vectors

Definition

A tangent vector is a 4-vector representing an infinitesimal displacement in spacetime.

- A tangent to the worldline of some observer has components given by the 4-velocity $u^\mu = (\gamma, \gamma\vec{v})$.
- In general relativity, only infinitesimal displacements are vectors.
- This is similar to the fact that the infinitesimal rotations are vectors, but large rotations are not (because rotations along different axes do not commute).

The Meaning of the Metric

The metric $\eta_{\mu\nu}$ gives a notion of "length in spacetime" to these tangent vectors u^μ by the formula

$$ds^2(u) = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} u^\mu u^\nu .$$

This formula is valid even in general relativity, although the matrix $\eta_{\mu\nu}$ will change.

Curves and Worldlines

Definition

A curve is called spacelike, timelike, or null, if the tangent to the curve at each spacetime point is spacelike, timelike, or null, respectively.

- A timelike curve is the worldline of massive observer and always has $\left|\frac{d\vec{x}}{dt}\right| < 1$.
- A null curve is the worldline of massless observer and always has $\left|\frac{d\vec{x}}{dt}\right| = 1$.
- A spacelike curve is not the worldline of any observer but represents a spatial displacement and has $\left|\frac{d\vec{x}}{dt}\right| > 1$.

The Physical Interpretation of "Length in Spacetime"

Definition

For a timelike curve, the proper time along the curve is given by

$$\tau = \int d\tau = \int \sqrt{-ds^2(u)} = \int \left(\sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} u^\mu u^\nu\right)^{1/2},$$

where u^μ is the tangent to the curve.

- To study special relativity in curvilinear coordinates, replace the $dx^2 + dy^2 + dz^2$ by the corresponding expression in our new coordinates.
- This defines a new metric $\eta'_{\mu\nu}$.
- The proper time between two events is given by the above formula, with $\eta'_{\mu\nu}$ taking the place of $\eta_{\mu\nu}$.

Special Relativity in Spherical Coordinates, I

Example

Consider spherical coordinates (r, θ, ϕ). The spacetime interval takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 = \sum_{\mu,\nu=0}^{3} \eta'_{\mu\nu} dx^\mu dx^\nu,$$

where x^μ are the coordinates (t, r, θ, ϕ) and

$$\eta'_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}.$$

Special Relativity in Spherical Coordinates, II

Example

Thus, the length of tangent vectors is given by

$$ds^2(u) = -(u^t)^2 + (u^r)^2 + r^2(u^\theta)^2 + r^2 sin^2\theta(u^\phi)^2.$$

In particular, if we have some worldline $(t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda))$, where λ is a parameter, then the proper time between two events is given by

$$\tau = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{(t')^2 - (r')^2 - r^2(\theta')^2 - r^2 sin^2\theta(\phi')^2}.$$

How Not to Write the Klein-Gordon Equation

Recall the we could write the Klein-Gordon equation as

$$\left(\sum_{\mu,\nu=0}^{3} \eta^{\mu\nu} \partial_\mu \partial_\nu - m^2 \right) \varphi = 0,$$

where $\eta^{\mu\nu}$ was the inverse metric. We might guess that the formula in curvilinear coordinates is

$$\left(\sum_{\mu,\nu=0}^{3} \eta'^{\mu\nu} \partial_\mu \partial_\nu - m^2 \right) \varphi \stackrel{?}{=} 0,$$

with $\eta'^{\mu\nu}$ being the inverse of $\eta'_{\mu\nu}$, but we would be wrong!

The Correct Generalization

Fact

The correct generalization is

$$(\Box - m^2)\varphi = \left(\sum_{\mu,\nu=0}^{3} \frac{1}{\sqrt{\eta'}} \partial_\mu \left(\eta'^{\mu\nu} \sqrt{\eta'} \partial_\nu \right) - m^2 \right) \varphi = 0,$$

where $\eta' = -\det \eta'_{\mu\nu}$.

- If the components of $\eta'_{\mu\nu}$ are constant, this reduces to our guess.
- The extra factors of η' are Jacobians which make the expression coordinate independent.

The Klein-Gordon Equation in Spherical Coordinates

Example

In spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

so that

$$\eta'^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}$$

and $\eta = r^4 \sin^2\theta$. Plugging into the above formula gives

$$\left(-\partial_t^2 + \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 - m^2\right)\varphi = 0.$$

Restating the Problem

The above equation can be written as

$$\partial_t^2 \varphi = \left(\nabla^2_{\mathbb{S}^2 \times \mathbb{R}} - m^2\right) \varphi$$

where $\nabla^2_{\mathbb{S}^2 \times \mathbb{R}}$ represents the Laplacian in spherical coordinates. We want to solve this equation, subject to initial conditions φ_0 and π_0. How?

The Traditional PDE Solution

1. Find the eigenfunctions of the Laplacian in spherical coordinates: $j_l(kr)$ and $Y_{l,m}(\theta, \phi)$.
2. Project the initial data onto the eigenfunctions:

$$c_{k,l,m} = \int_{\mathbb{R}^3} \varphi_0(r,\theta,\phi) j_l(kr) Y^*_{l,m}(\theta,\phi) r^2 \sin\theta \, dr \, d\theta \, d\phi,$$

$$d_{k,l,m} = \int_{\mathbb{R}^3} \pi_0(r,\theta,\phi) j_l(kr) Y^*_{l,m}(\theta,\phi) r^2 \sin\theta \, dr \, d\theta \, d\phi.$$

3. Sum over the modes, evolving each one independently:

$$\varphi(\vec{x}, t) = \int_0^\infty dk \sum_{l,m} \left(c_{k,l,m} \cos(\omega t) + d_{k,l,m} \frac{\sin(\omega t)}{\omega} \right) j_l(kr) Y_{l,m}(\theta,\phi),$$

where $\omega = \sqrt{k^2 + m^2}$.

A Comparison

In spherical coordinates, the solution is:

$$\varphi(\vec{x}, t) = \int_0^\infty dk \sum_{l,m} \left(c_{k,l,m} \cos(\omega t) + d_{k,l,m} \frac{\sin(\omega t)}{\omega} \right) j_l(kr) Y_{l,m}(\theta, \phi),$$

In Cartesian coordinates, the solution was:

$$\varphi(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(\hat{\varphi}_0(\vec{k}) \cos \omega t + \hat{\pi}_0(\vec{k}) \frac{\sin \omega t}{\omega} \right) e^{i\vec{k}\cdot\vec{x}} d^3 k$$

These are of the exact same form!

The Operator Viewpoint

A second way is to view the Laplacian as an operator and write the solution

$$\begin{aligned}\varphi(\vec{x}, t) &= \cos\left[\left(\nabla^2_{\mathbb{S}^2\times\mathbb{R}} - m^2\right)^{1/2} t\right]\varphi_0 \\ &\quad + \left(\nabla^2_{\mathbb{S}^2\times\mathbb{R}} - m^2\right)^{-1/2} \sin\left[\left(\nabla^2_{\mathbb{S}^2\times\mathbb{R}} - m^2\right)^{1/2} t\right]\pi_0.\end{aligned}$$

In this section, we will figure out what this expression means, and reinterpret our previous solutions as special cases of this equation.

Functions of Matrices

Let A be an $n \times n$ matrix.

- $f(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} A^k$.
- If A is a real symmetric matrix, it is orthogonally diagonalizable: $A = P^{-1}DP$. Then

$$f(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} P^{-1} D^k P.$$

- This is much easier to compute.

Projections

Let A be symmetric and apply it to some vector v. The result can be written in the form

$$Av = \sum_a aP_a(v) := \sum_a a \left|a\right\rangle \left\langle a \mid v\right\rangle .$$

The operators $P_a = \left|a\right\rangle \left\langle a\right|$ are called the projection operators or projectors of A. This identity follows from the rules of matrix multiplication and the following two facts:

1. The rows/columns of P and P^{-1} are the eigenvectors of A.
2. The diagonal of D consists of the eigenvalues of A

Functions of Matrices Via Projectors

Theorem

(Spectral Theorem for Real Symmetric Matrices)

If A is a real, symmetric $n \times n$ matrix, then

$$f(A) = \sum_a f(a) |a\rangle \langle a| .$$

In particular,

$$1_{n\times n} = \sum_a |a\rangle \langle a| .$$

Functions of Operators

Theorem

(Spectral Theorem for Operators)

If L is a linear, self-adjoint operator on a Hilbert space, then it has a complete set of eigenvectors and

$$f(L) = \sum_{\lambda} f(\lambda) |\lambda\rangle \langle\lambda| .$$

- As before, $1 = \sum_{\lambda} |\lambda\rangle \langle\lambda| .$
- For the Hamiltonian H, we have

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle = \sum_{E} e^{-iEt} |E\rangle \langle E \mid \psi(0)\rangle .$$

Back to Klein-Gordon, I

- The Laplacian is self-adjoint (it is the Hamiltonian for a free particle).
- $\triangle = \nabla^2_{\mathbb{S}^2\times\mathbb{R}} - m^2$ is also self-adjoint.
- We want to compute

$$\varphi(\vec{x}, t) = \cos\left(\triangle^{1/2} t\right)\varphi_0 + \triangle^{-1/2}\sin\left(\triangle^{1/2} t\right)\pi_0$$

Back to Klein-Gordon, II

To compute, we use the spectral theorem.

1. Find its eigenfunctions: $|k,l,m\rangle = j_l(kr)Y_{l,m}$.
2. Project the initial data onto the eigenfunctions

$$\langle k,l,m \,|\, \varphi_0\rangle = c_{k,l,m} = \int_{\mathbb{R}^3} \varphi_0(r,\theta,\phi) j_l(kr) Y^*_{l,m}(\theta,\phi) r^2 \sin\theta \, dr \, d\theta \, d\phi,$$

$$\langle k,l,m \,|\, \pi_0\rangle = d_{k,l,m} = \int_{\mathbb{R}^3} \pi_0(r,\theta,\phi) j_l(kr) Y^*_{l,m}(\theta,\phi) r^2 \sin\theta \, dr \, d\theta \, d\phi.$$

3. Apply the function to the eigenvalue of each mode independently.

Back to Klein-Gordon, III

Putting it all together:

$$\begin{aligned}
\varphi(\vec{x},t) &= \cos\left(\triangle^{1/2}t\right)\varphi_0 + \triangle^{-1/2}\sin\left(\triangle^{1/2}t\right)\pi_0 \\
&= \int_0^\infty dk \sum_{l,m}\Bigg(\cos\left(\sqrt{k^2+m^2}\,t\right)|k,l,m\rangle\langle k,l,m\,|\,\varphi_0\rangle \\
&\qquad + \frac{\sin\left(\sqrt{k^2+m^2}\,t\right)}{\sqrt{k^2+m^2}}|k,l,m\rangle\langle k,l,m\,|\,\pi_0\rangle\Bigg) \\
&= \int_0^\infty dk \sum_{l,m}\left(c_{k,l,m}\cos(\omega t) + d_{k,l,m}\frac{\sin(\omega t)}{\omega}\right) j_l(kr)Y_{l,m}(\theta,\phi)
\end{aligned}$$

This is exactly the same as in the PDE solution!

Operators: The Good and the Bad

The operator viewpoint allows us to write down solutions without computing the eigenfunctions of the Laplacian. This is both good and bad.

- Since we can write a formula without referring to the specific coordinates, we can find a solution which applies to many different spacetimes simultaneously.
- Since we won't actually find the eigenfunctions, we will need an indirect argument to show that they exist.

General Relativity

- In general relativity, matter causes spacetime to become curved.
- We call the metric $g_{\mu\nu}$, reserving $\eta_{\mu\nu}$ for the flat metric.
- "Curvature" means that the metric, $g_{\mu\nu}$, varies from point to point in spacetime.
- How do we know if the metric is spacetime-dependent, and how does the matter cause this?

Curvature

- There are different measures of curvature: Riemann tensor ${R_{\alpha\beta\gamma}}^{\delta}$, Ricci tensor $R_{\mu\nu}$, Einstein tensor $G_{\mu\nu}$.
- All are defined by formulae of the form:

curvature = linear combinations of second order derivatives of the metric

- For the metric of special relativity, all of the curvature tensors are zero in all coordinate systems!
- A spacetime is curved provided at least one tensor has a non-zero component.

Stress Tensor and Einstein's Equations

- Every matter field has a stress tensor, $T_{\mu\nu}$, associated to it.
- T_{00} is the energy density, T_{0i} is its spatial momentum, and T_{ij} are the stresses in the field.
- Einstein's equation relates the stress tensor to the Einstein curvature tensor:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

The Klein-Gordon Equation in Curved Spacetime

Fact

The generalization of Klein-Gordon to general relativity is

$$(\Box - m^2)\varphi = \left(\sum_{\mu,\nu=0}^{3} \frac{1}{\sqrt{g}} \partial_\mu \left(g^{\mu\nu} \sqrt{g} \partial_\nu \right) - m^2 \right) \varphi = 0,$$

where $g = -\det g_{\mu\nu}$.

Curves in General Relativity

Definition

A curve is called spacelike, timelike, or null, if the tangent to the curve at each spacetime point is spacelike, timelike, or null, respectively.

- The same definition as in special relativity
- The three types have the same interpretation.

Causal Curves in General Relativity

Definition

A causal curve is a curve which is timelike or null at each point.

- In general relativity, the notion of velocity is subtle due to spacetime curvature.
- Information flow along causal curves is the precise meaning of "nothing can move faster than light".

Cauchy Horizons in General Relativity

Definition

A spatial surface is a 3-dimensional subset of spacetime composed of spacelike curves.

Definition

Given a spatial surface S, let A be the set of points from which you can follow a causal curve and still avoid hitting the surface S. The boundary of the set A is called the Cauchy horizon of S.

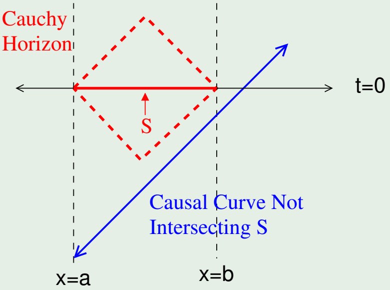
The Line Segment Revisited

Example

S is the plate $a \leq x \leq b$, y and z arbitrary on the t-axis. Clearly, any curve inside the surface will have

$$\left|\frac{d\vec{x}}{dt}\right| = \frac{|\vec{x}|}{0} = \infty > 1.$$

Therefore, S is spatial.



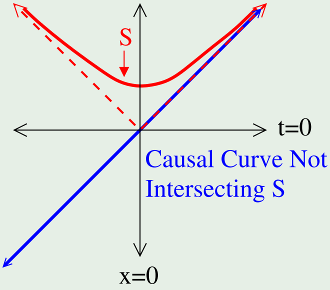
The Hyperbola Revisited

Example

S is the hyperbola $t^2 - x^2 = 1$, $t > 0$. The slope at the point $(x, \sqrt{x^2+1})$ is $\frac{dt}{dx} = \frac{x}{\sqrt{x^2+1}}$, or

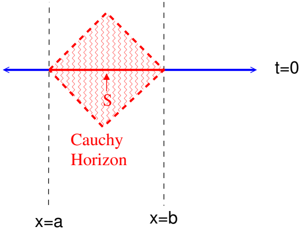
$$\frac{dx}{dt} = \frac{\sqrt{x^2+1}}{x} > 1.$$

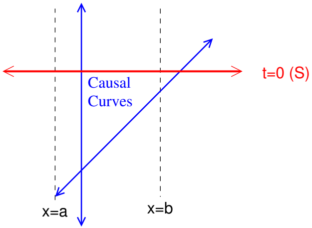
Therefore the curve is spacelike and the surface (created by translating the curve in the y and z directions) is spatial.



Information Flow

- Recall that causal curves represent possible information flow.
- Within the Cauchy horizon, all the information came from S, but outside information came from elsewhere.
- To find a global solution, need to specify initial conditions on a surface without a Cauchy horizon.





Operators to the Rescue!

Definition

A spacetime is non-globally-hyperbolic spacetimes if the metric is such that every spatial surface has a Cauchy horizon.

- Need some way to extend solutions past the horizon.
- Unlike flat space, there is no guarantee that an extension exists.
- However, operators will allow us to find extensions!

Solving the Problem

- Consider stationary spacetimes (those with time translation symmetry).
- Using Hamiltonian mechanics, we can rewrite the Klein-Gordon equation in first order in time form:

$$\partial_t \Phi = -h\Phi,$$

  where $\Phi = (\varphi, \pi)$, and h is 2×2 matrix built from spatial derivatives and the inverse metric.
- If the operator h has a spectral decomposition, we can write down the solution immediately:

$$\Phi(t) = e^{-ht}\Phi_0.$$

What Needs to Be Checked?

- Does h have a complete set of eigenfunctions?
- Is the extended solution an honest solution of the equation?
- Is the solution non-chaotic?

Proven Properties of Solutions

Theorem

(Seggev, 2004) Consider a stationary spacetime, and a slicing of that spacetime into spatial surfaces obeying a mild geometric condition. Then

1. *the operator h has at least one complete set of eigenfunctions;*
2. *functions of the form*

$$\Phi(\vec{x}, t) = e^{-ht}\Phi_0(\vec{x})$$

   are global solutions of the Klein-Gordon equation;
3. *the solutions are non-chaotic; and*
4. *energy is automatically conserved.*

Field Quantization in Flat Space

- To quantize the Klein-Gordon equation in flat space, we promote the Fourier coefficients to operators:

$$\varphi(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{1}{2\omega} \left(\hat{a}(k) e^{i(\omega t - \vec{k}\cdot\vec{x})} + \hat{a}(k)^\dagger e^{-i(\omega t - \vec{k}\cdot\vec{x})} \right) d^3k$$

- $\hat{a}^\dagger(k)$ and $\hat{a}(k)$ are creation and annihilation operator, respectively, and obey $[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}')$.
- The association of operators to specific modes is called a complex structure.
- The above complex structure is dictated by Poincaré invariance.

Field Quantization in Curved Space

- Solving the initial value problem suffices to find the eigenmodes, but still need a complex structure (no Lorentz invariance).
- $|h|^{-1}h$ is a complex structure!
- Generalization of "frequency splitting" quantization to non-globally-hyperbolic spacetimes.
- Interacting fields have difficulties, even at tree (classical) level.

Conclusions and Future Directions

- Certain astronomically or theoretically interesting spacetimes are non-globally-hyperbolic; such spacetimes may not have solutions to the Klein-Gordon equation.
- I have shown that such spacetimes, if stationary, do have solutions.
- Operators are a useful tool in the free case, but they seem limited in the interacting case.
- What about the Dirac equation?